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Short- and long-time behavior of aquifer drainage after slow and sudden recharge according to the linearized Laplace equation

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Abstract

During the last 25 years there has been a great interest in deriving aquifer characteristics from outflow data. This analysis has been mainly based of the drainage of a horizontal aquifer after sudden drawdown, using the Boussinesq approximation. Following the general approach of Brutsaert and Lopez [Brutsaert W, Lopez, JP. Basin-scale geohydrologic drought flow features of riparian aquifers in the southern Great Plains. Water Resour Res 1998;34(2):233–40], it was determined that for this geometry the aquifer behavior could be characterized by $\mathrm{d}Q/\mathrm{d}t \propto Q^3$ for small t and by $\mathrm{d}Q/\mathrm{d}t \propto Q^{3/2}$ for large t. It was remarked that $\mathrm{d}Q/\mathrm{d}t \propto Q$ for large t is often observed. In practice, it is also difficult to determine if $\mathrm{d}Q/\mathrm{d}t \propto Q^3$ for small t because this behavior can only be observed over a very short period.

Here, we present a similar analysis of aquifer behavior based on the more fundamental Laplace solution for penetrated aquifers. It has been shown that also when the drain does not fully penetrate the aquifer, the solution still produces good results [Szilagyi, J. Sensitivity analysis of aquifer parameter estimations based on the Laplace equation with linearized boundary conditions. Water Resour Res 2003;39(6)]. The Laplace solution quickly shows that $\mathrm{d}Q/\mathrm{d}t \propto Q$ for $t \to \infty$ and $\mathrm{d}Q/\mathrm{d}t \propto Q^{\infty}$ for $t \to 0$, after sudden drawdown. This analysis reconfirms previous findings concerning long-time behavior. More importantly, the analysis shows that the exponent B in $\mathrm{d}Q/\mathrm{d}t \propto Q^B$ does not have a fixed limited value for short times for the given geometry. Further analysis, however, shows that under certain conditions the relation $\mathrm{d}Q/\mathrm{d}t \propto Q^3$ is retained for $0 \ll t < 1$. Detailed examination of the Laplace solution also shows under which types of recharge dynamics a well-identifiable transition takes place between short- and long-term behavior. As long as such a clear transition exists, the aquifer characterization method proposed earlier by Brutsaert and Lopez [Brutsaert W, Lopez, JP. Basin-scale geohydrologic drought flow features of riparian aquifers in the southern Great Plains. Water Resour Res 1998;34(2):233–40] can be applied. It is shown that for a sharp pulse input, the Laplace solution gives similar results as presented by Brutsaert and Lopez [Brutsaert W, Lopez, JP. Basin-scale geohydrologic drought flow features of riparian aquifers in the southern Great Plains. Water Resour Res 1998;34(2):233–40]. For a smooth pulse, the transition becomes unclear. What is "smooth" and "sharp" depends on input and aquifer characteristics, whereby shallow aquifers give clearer transitions than deep aquifers for the same input. The analysis shows that when rain ceases suddenly after the aquifer has come into equilibrium with a steady rain input, a usable transition in the relation between $\mathrm{d}Q/\mathrm{d}t$ and Q can be found as well. Researchers can use the present analysis to assess whether specific aquifers and recharge events can be used for the previously suggested characterization method.

Keywords: Laplace equation; Boussinesq equation; Aquifer parameter estimation; Drought flow analysis

1. Introduction

Recession flow analysis is an important tool for aquifer characterization, especially under conditions where

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reliable direct parameter and/or recharge measurements are not available or insufficient. Dewandel et al. [3], for example, evaluated many different conceptual recession models to quantify the thickness of an aquifer in Oman. Similarly, Wittenberg and Sivapalan [10], did not have sufficient high quality recharge data and, therefore, used a single non-linear reservoir to describe recession from the aquifer that feeds the Harris River in Western Australia. Whereas Wittenberg and Sivapalan [10] used a fixed exponential to describe the relation between storage and discharge, it has been found elsewhere that exponentials may vary over time. Specifically, it has been pointed out that aquifers discharge differently immediately after a recharge event and after longer times [2,6,5]. Different behavior for short and long times may actually provide extra information on the aquifer parameters. The present article contributes mainly to the latter approach, as put forward by Brutsaert and Lopez [2]. In its simplest form, this approach may be summarized as describing the emptying of an aquifer after recharge with

$$-\mathrm{d}Q/\mathrm{d}t = AQ^B \tag{1}$$

with Q discharge (L^3/T or L^2/T), t as time, and A and B model parameters. By plotting $\ln(-\mathrm{d}Q/\mathrm{d}t)$ versus $\ln(Q)$ one can obtain the model parameters, which in turn may be linked to parameters such as aquifer thickness, hydraulic conductivity and specific storage or drainable porosity. In order to link the parameters from Eq. (1) to physically interpretable parameters, different groundwater flow models can be used. The groundwater model used most often is the Boussinesq equation which is solved for short and long times after sudden drawdown, with zero depth in the drains for $t > 0$, see Fig. 1B [2,8,6,5].

A general solution for aquifer drainage using the Laplace equation for the geometry given in Fig. 1A is given by van de Giesen et al. [9]. In general, the Laplace equation describes two-dimensional groundwater flow, while Boussinesq reduces the flow to a one-dimensional phenomenon. The Laplace equation is, therefore, more fundamental and for the geometry in Fig. 1A it can be shown that the Boussinesq solution is a special limiting case of the solution under the Laplace equation. The main advantage of the solution put forward by van de Giesen et al. [9] is that it can be used for arbitrary initial conditions and recharge patterns.

In this article, the Laplace solution is used is used primarily to examine whether the aquifer with the geometry of Fig. 1A behaves like the aquifer in Fig. 1B, as when the previously reported upon Boussinesq-based approach were followed. Because the Laplace-based solution is flexible, it can also be used to examine the effects of the exact recharge dynamics. In the literature, a physically implausible sudden drawdown is normally used to find the Boussinesq solution. Here, the Laplace solution is used to see how fast such a drawdown needs to be in order to produce a sharp transition between short- and long-time behavior. In turn, this can be used to refine recession flow analysis based on Eq. (1).

The main contribution of this article is an analytical method to examine under which circumstances the approach put forward by Brutsaert and Lopez [2] can be used to derive aquifer characteristics such as conductivity and drainable porosity. Because the remainder is somewhat abstract if one is not familiar with Brutsaert and Lopez [2], their main approach is repeated here for the reader's convenience. First, one looks at actual stream flow measurements when the flow recedes after rain stops. When Q_i and Q_{i+1} are two measurement Δt time apart, $\mathrm{d}Q/\mathrm{d}t$ is approximated by $(Q_i - Q_{i+1})/\Delta t$ and $Q(t)$ by $(Q_i + Q_{i+1})/2$. These approximate values are plotted against each other in a log–log plot. Fig. 2 shows a sketch of results that are typically obtained. Because the aquifer is assumed to be the slowest reacting element in the watershed, one now takes the outer envelopes to fit lines for short times ($B = 3$) and long times ($B = 1$). These lines are now used to calculate the logarithmic of A in Eq. (1). The $A_{B=1}$, associated with $B = 1$ and long times, is according to Brutsaert and Lopez [2]

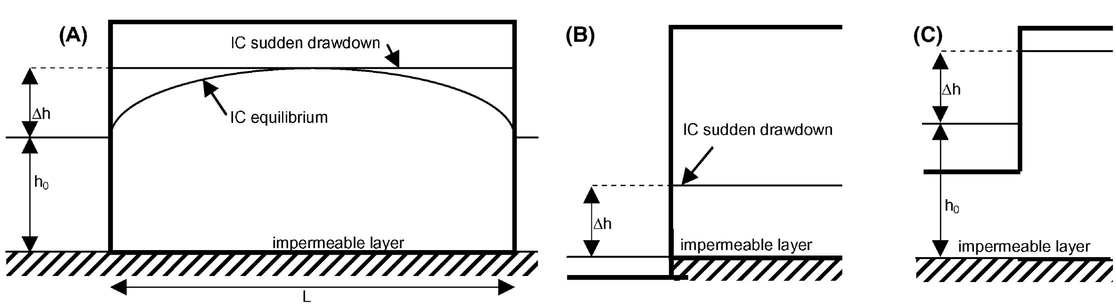

Fig. 1. Geometry and different boundary (BC) and initial conditions (IC) discussed. (A) Drains fully penetrate aquifer, water depth in drains large compared to changes in water table (main geometry discussed). (B) Same as (A) but with zero water height in the drains after drawdown. (C) Same as (A) but with partially penetrating drain.

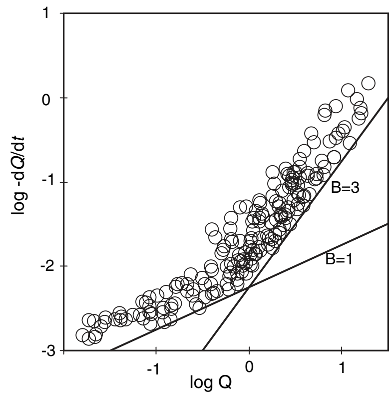

Fig. 2. Sketch of typical results found by Brutsaert and Lopez [2], with Q and t in arbitrary units. The fixed lines show the outer envelopes for two values of the exponent B in Eq. (1).

$$A_{B=1} = 0.3465 \cdot \pi^2 \cdot K \cdot D \cdot U^2/(\mu \cdot W^2) \tag{2}$$

where K is the saturated hydraulic conductivity, D is the aquifer thickness, U is the total length of upstream channels in the watershed, μ is the drainable porosity, and W is the area of the watershed. Similarly, $A_{B=3}$, associated with $B = 3$ and short times is

$$A_{B=3} = 1.1334/(K \cdot \mu \cdot D^3 \cdot U^2) \tag{3}$$

By combining Eqs. (2) and (3), one can calculate K and μ, after deriving the values for U and W from a topographic map, and after collecting a reasonably reliable estimate of D from, for example, bore-logs.

Because the points in the $\ln(-\mathrm{d}Q/\mathrm{d}t)$ versus $\ln(Q)$ graph tend to have quite some scatter, it is not always clear where to draw the envelope lines. Part of the scatter is due to the fact that not only the aquifer but also other watershed elements may contribute to the recession flow. In this article, it will be shown that even when the contribution of such elements is negligible, large scatter may still be observed. This remaining scatter is due to the fact that not every recession period will produce a clear transition between short- and long-time behavior. Some recessions will behave differently and increase the scatter among the data points. By filtering out such recession periods, the analysis proposed by Brutsaert and Lopez [2] can be applied to data with less scatter. In the following, it will be shown how exactly different recharge dynamics affect the $\ln(-\mathrm{d}Q/\mathrm{d}t)$ versus $\ln(Q)$ graph.

2. Model

The notations and the solution of the Laplace equation as given in van de Giesen et al. [9] are followed, together with the notation. This solution was given in the form of a very general Green's function and only a few applications with explicit solutions were provided. Although the corresponding geometry as given in Fig. 1A may look very specific and somewhat artificial, it represents one of the classic cases for which drainage problems are typically solved. Furthermore, it has been suggested by numerical simulations that it is not necessary that the drains fully penetrate the aquifer and that the solution at hand may also be valid for partially penetrated aquifers as in Fig. 1C [7].

Because we seek to investigate further the aquifer recession behavior following the method presented in Brutsaert and Lopez [2], we are specifically interested in the shape of the $\ln(-\mathrm{d}Q/\mathrm{d}t)$ versus $\ln(Q)$ curves, which would yield the B exponent in Eq. (1). For these purposes, the interest lies on evaluation of the discharge and its time-derivative, even though the general solution also allows for evaluation of the water table position and streamline patterns. First, the original Green's function is evaluated for several cases to give explicit analytical solutions for Q and $\mathrm{d}Q/\mathrm{d}t$. These solutions are also evaluated at $t = 0$ and $t \to \infty$. The first case concerns sudden drawdown, which is equivalent to an impulse input, and inputs that rise and fall more slowly. The second case is recession following cessation of recharge after the water table has come into equilibrium with a constant recharge rate. It is thought that these two cases, which will both be examined for fast and slow changes, are extremes that would bracket any real recession events. In Section 3 the analytical solutions will be used to calculate behavior for intermediate times.

2.1. Sudden drawdown and impulse inputs

The discharge per unit width from a fully penetrated aquifer after sudden drawdown is given by

$$Q(t) = \sum_{n=1,3,\ldots}^{\infty} K\frac{4\Delta h}{n\pi}\tanh\left(\frac{n\pi h_0}{L}\right) \cdot \exp\left(-\frac{K}{\mu}\tanh\left(\frac{n\pi h_0}{L}\right)\frac{n\pi}{L}t\right) \tag{4}$$

where K is the saturated hydraulic conductivity (m^2/s), Δh the drawdown at $t = 0$ (m), h_0 the depth of water above the impermeable layer after drawdown in the drain or river (m), L the distance between parallel drains (m), and μ the drainable porosity (–). It is convenient to make t and Q dimensionless with

$$t^* = \frac{K}{\mu L}t, \quad Q^* = \frac{Q}{K \cdot L}, \quad h_0^* = h_0/L \quad \text{and} \quad \Delta h^* = \Delta h/L \tag{5}$$

which gives

$$Q^*(t^*) = \sum_{n=1,3,\ldots}^{\infty} \frac{4\Delta h^*}{n\pi}\tanh(n\pi h_0^*) \cdot \exp(-\tanh(n\pi h_0^*) \cdot n\pi \cdot t^*) \tag{6}$$

and

$$dQ^*/dt^* = -\sum_{n=1,3,\ldots}^{\infty} 4\Delta h^*(\tanh(n\pi h_0^*))^2 \cdot \exp(-\tanh(n\pi h_0^*) \cdot n\pi \cdot t^*) \quad (7)$$

For $t^* \to \infty$, the first term in these series dominates, which can be shown by referring to the fact that $0 < \tanh(n\pi h_{0^*}) < 1$, and by proving that for any finite $\Gamma > 1$ we can find a sufficiently large t^* such that

$$\exp(-t^*) > \Gamma * \sum_{n=3,5,\ldots}^{\infty} \exp(-n \cdot t^*) \quad (8)$$

Because for a fixed t^*

$$\Gamma * \sum_{n=3,5,\ldots}^{\infty} \exp(-n \cdot t^*) < \frac{\Gamma}{2} * \int_2^{\infty} \exp(-t^* \cdot \upsilon)d\upsilon = \frac{\Gamma}{2t^*} \exp(-2 \cdot t^*) \quad (9)$$

one sees that Eq. (8) is true whenever $2t^* \cdot \exp(2t^*) > \Gamma$. By retaining the first terms for $t^* \to \infty$ one now obtains

$$\lim_{t^* \to \infty} \ln(-dQ^*/dt^*) = \ln(Q^*) + \ln(\pi \cdot \tanh(\pi h_0^*)) \quad (10)$$

Therefore, the slope in the $\ln(-dQ^*/dt^*)$ versus $\ln(Q^*)$ graph goes to one for long times. This is similar to one of the solutions based on the Boussinesq equation for the boundary condition $h_0 = 0$ presented by Brutsaert and Lopez [2], which also used a linearization. It should be stressed that at long times, the aquifer seems to behave like a linear reservoir because we first linearized the boundary conditions. If such a linearization cannot be justified, one should expect different results. Parlange et al. [5] solved the Boussinesq equation after complete drawdown (Fig. 1B), in which case the linearization used here is not valid, and found a slope of 3/2 for the $\ln(-dQ^*/dt^*)$ versus $\ln(Q^*)$ graph for long times.

For the short time solution, we first remark that both Q^* and $-dQ^*/dt^*$ go to infinity and that

$$Q^* < \sum_{n=1,3,\ldots}^{\infty} 1/n \cdot \exp(-\tanh(\pi h_0^*) \cdot n \cdot t^*) \quad (11)$$

and

$$dQ^*/dt^* > \tanh(\pi h_0^*) \sum_{n=1,3,\ldots}^{\infty} \exp(-n \cdot t^*) \quad (12)$$

For the slope of $\ln(-dQ^*/dt^*)$ versus $\ln(Q^*)$ for $t \to 0$ we see

$$\lim_{t \to 0} \frac{\ln(dQ^*/dt^*)}{\ln(Q^*)} > \lim_{N \to \infty} \frac{\ln(\tanh(\pi h_0^*) \cdot N)}{\ln(\ln(N))} = \infty \quad (13)$$

So, there is no limiting slope for small times in the case of a sudden drawdown.

Looking at Fig. 1A, one sees that a sudden drawdown is achieved when at $t < 0$ the water table in the drains has the same height, $h_0 + \Delta h$, as the groundwater in the aquifer, and when at $t = 0$, the water level in the drains is instantaneously lowered to h_0. For $t = 0$, we arrive at the same initial conditions if we start out with a water level of h_0 in drains and aquifer, and at $t = 0$ instantaneously raise the groundwater level in the aquifer to $h_0 + \Delta h$. Such a rise in groundwater level would be achieved by an impulse input. Therefore, the reaction of the aquifer after a sudden drawdown is the same as after an instantaneous impulse input. It would be interesting to see what the effect is of a more gently rising and falling input, for example a cosine shaped input

$$P^*(x,t^*) = \begin{cases} 1/2P^*\lambda \cdot (1 - \cos(\lambda\pi t^*)) \\ \quad \text{for } 0 < t^* < 2/\lambda \\ 0 \quad \text{for } t^* > 2/\lambda \end{cases} \quad (14)$$

with $P^* = \frac{P}{K}$. For $\lambda \to \infty$, we have an instantaneous impulse input again with a volume P^*. The solution for this input, when starting out with a flat water table, is

$$Q^*(t^*) = \begin{cases} \sum_{n=1,3,\ldots}^{\infty} \frac{2P^*\lambda}{n^2\pi^2} \cdot \left\{1 - \frac{\lambda^2\pi^2}{\beta_n^2+\lambda^2\pi^2}\exp(-\beta_n t^*) - \frac{\beta_n^2}{\beta_n^2+\lambda^2\pi^2}\cos(\lambda\pi t^*) - \frac{\beta_n \cdot \lambda\pi}{\beta_n^2+\lambda^2\pi^2}\sin(\lambda\pi t^*)\right\} & 0 < t^* < 2/\lambda \\ \sum_{n=1,3,\ldots}^{\infty} \frac{2P^*\lambda}{n^2\pi^2} \cdot \left\{1 - \frac{\lambda^2\pi^2}{\beta_n^2+\lambda^2\pi^2}\exp(-2\beta_n/\lambda) - \frac{\beta_n^2}{\beta_n^2+\lambda^2\pi^2}\right\} \cdot \exp(-\beta_n(t^* - 2/\lambda)) & t^* > 2/\lambda \end{cases} \quad (15)$$

Because this input shape has a continuous derivative, both Q^* and dQ^*/dt^* are finite for $t^* = 0$.

2.2. Drainage after equilibrium

Above, we examined pulse-type inputs that assume a flat water table at $t^* = 0$. In order to cover a completely different input type, we look at what happens when rain or recharge stops after the water table has come into equilibrium with a constant recharge rate. First, we will briefly look at what happens if rain or recharge simply ceases after the water levels in the aquifer have come into equilibrium with a constant rainfall rate. Second, we will look at the short time behavior if rain or recharge ceases gradually after the aquifer levels have come into equilibrium with a constant rainfall rate. Probably the latter case is physically more realistic.

If, starting with a flat water table, it rains for a sufficiently long time ($t > t_{eq}$), an equilibrium profile is attained when the recharge equals the aquifer discharge ($Q(t) = 1/2PL$). When the rain ceases after attaining equilibrium, we can use the general Green's function [9] to calculate

$$Q(t^*) = \sum_{n=1,3,\ldots}^{\infty} \frac{4P^*}{n^2\pi^2} \cdot \exp(-\tanh(n\pi h_0^*) \cdot n\pi \cdot t^*) \qquad (16)$$

and

$$-\mathrm{d}Q^*/\mathrm{d}t^* = \sum_{n=1,3,\ldots}^{\infty} \frac{4P^*}{n\pi} \cdot \tanh(n\pi h_0^*) \cdot \exp(-\tanh(n\pi h_0^*) \cdot n\pi \cdot t^*) \qquad (17)$$

For short times, $\mathrm{d}Q^*/\mathrm{d}t^*$ does not converge, whereas Q^* is finite ($\pi^2 = \sum_{n=1,3,\ldots}^{\infty} 8/n^2$). So, again, the slope of $\ln(-\mathrm{d}Q^*/\mathrm{d}t^*)$ versus $\ln(Q^*)$ goes to infinity for short times. This short time behavior may mainly be due to the physically implausible assumptions of instantaneous drawdown (see Section 2.1) or, here, the abrupt halting of recharge. It may, therefore, be worthwhile to see what happens if there is a smooth transition from an equilibrium state to zero recharge. The generality of the linearized Laplace solution basically allows for such a transition to have any arbitrary form. Here, we assume a cosine-shaped decline in rainfall from P to zero after equilibrium has been reached ($t_{\mathrm{eq}} = 0$)

$$P^*(x,t^*) = \begin{cases} P^* & \text{for } t^* < 0 \\ 1/2P^* \cdot (\cos(\lambda\pi t^*) + 1) & \text{for } 0 < t^* < 1/\lambda \\ 0 & \text{for } t^* > 1/\lambda \end{cases} \qquad (18)$$

One reason for choosing a cosine-shaped decline is that it resembles the end of the rainy season in West Africa, where the authors pursue part of their hydrological research. Derivation of the solution is straightforward but tedious and results in the relatively complex expression:

$$Q^*(t^*) = \begin{cases} \sum_{n=1,3,\ldots}^{\infty} \frac{2P^*}{n^2\pi^2} + \frac{2P^*}{n^2\pi^2} \cdot \frac{1}{\beta_n^2+\lambda^2\pi^2} (\lambda^2\pi^2 \exp(-\beta_n t^*) + \beta_n^2 \cos(\lambda\pi t^*) + \beta_n \lambda\pi \sin(\lambda\pi t^*)) & 0 < t^* < 1/\lambda \\ \sum_{n=1,3,\ldots}^{\infty} \frac{2P^*}{n^2\pi^2} \cdot \frac{\lambda^2\pi^2}{\beta_n^2+\lambda^2\pi^2} (\exp(\beta_n/\lambda) + 1) \cdot \exp(-\beta_n t^*) & t^* > 1/\lambda \end{cases} \qquad (19)$$

with $\beta_n = \tanh(n\pi h_0^*) \cdot n\pi$. For long times, the aquifer behaves similarly as for the previous transitions. Not too surprisingly, we see that for $t = 0$, $\mathrm{d}Q^*/\mathrm{d}t^* = 0$. In the $\ln(-\mathrm{d}Q^*/\mathrm{d}t^*)$ versus $\ln(Q^*)$ graph, $\ln(-\mathrm{d}Q^*/\mathrm{d}t^*)$ would go to minus infinity for (very) short times. From the three examples given, we may conclude that there is not a clear-cut early time behavior of fully penetrated aquifers. Instead, any slope can be expected for $t = 0$ and the value depends on the abruptness with which the input ceases and on the initial conditions.

In the following, the above equations are used to examine the continuous behavior between $t = 0$ an $t \to \infty$. The reason is to see if there is, perhaps, intermediate behavior that depends less on the exact transition and could be used to determine aquifer parameters.

3. Results of Laplace solution

The solutions above show that for all initial conditions and input transitions, the slopes of $\ln(-\mathrm{d}Q^*/\mathrm{d}t^*)$ versus $\ln(Q^*)$ go to one for long times. Due to the linearization of the boundary conditions, the long-time response gives the classical extinction coefficient

$$Q(t) \propto \exp\left(-\frac{K\pi}{\mu L} \tanh\left(\frac{\pi h_0}{L}\right) \cdot t\right) \qquad (20)$$

The short time behavior very much depends on the type of transition. Based on the above analysis, any value between minus and plus infinity may be found. This stands in contrast with the findings for the boundary conditions examined previously which found a slope of three for short times. More importantly, especially in Brutsaert and Lopez [2] and Szilagyi [6], a very sharp transition was found between the two slopes that allowed for fairly robust fitting and parameter estimation. Here, we examine if for the present boundary conditions a similarly well-defined transition can be found.

3.1. Sudden drawdown and impulse inputs

For $t^* \ll 1$, Eqs. (6) and (7) converge very slowly which limits to some extent calculations for very short times. The terms in these equations, however, change very smoothly so it generally suffices to calculate only every mth term and calculate the sum of the intermediate terms through interpolation. In this way, a simple spreadsheet can be used to calculate the results presented here. A small value of Δh with respect to h_0^* was chosen to ensure that the linearization would not cause significant errors.

Fig. 3A shows the $\ln(-\mathrm{d}Q^*/\mathrm{d}t^*)$ versus $\ln(Q^*)$ graph for points from $t^* = 0.000005$ to $t^* = 80$. Please note that here, and in all following figures, time increases from right to left along the curves. Interestingly, different domains of behavior stand out with sharp transitions between them. This looks very much like the behavior previously reported in Brutsaert and Lopez [2]. For long

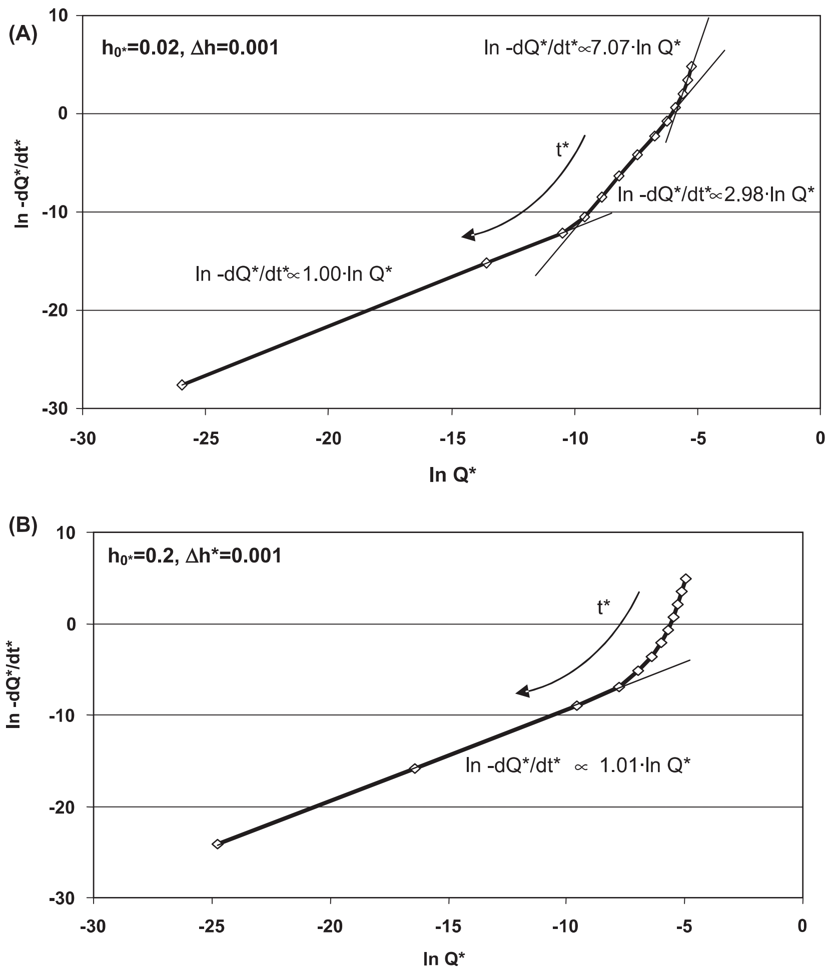

Fig. 3. (A) Sudden drawdown with $h_0^* = 0.02$. (B) Sudden drawdown with $h_0^* = 0.2$. Formulas refer to thin regression lines that are fitted to (more or less) linear segments. The right side of the curves corresponds to short times, the left side corresponds to long times.

times ($t^* > 1$), the slope of the graph is exactly one. Around $t^* = 1$, there is a sharp transition before which we see a slope of (almost) three. This behavior is exhibited from $t^* = 0.0003$ until $t^* = 1$. What is new, however, is that this slope of three does not continue for even shorter times but rises after yet another sharp transition to a slope of about seven. As shown above, the slope as $t \to 0$ should go to infinity which is consistent with the findings here. For practically measurable short times, however, it looks like one can safely assume a slope of three.

Revisiting Eq. (6), one sees that for very short times, the exponentials are close to one for the first terms in the series. If h_0^* is sufficiently small ($<\varepsilon \approx 0.02$), the first term becomes negligible ($<\varepsilon^2$) so $\mathrm{d}Q^*/\mathrm{d}t^* \propto Q^{*3}$. The fact that h_0^* needs to be small, suggests that the observed behavior depends on the parameterization. This parameter dependence becomes very clear from Fig. 3B, where $h_0^* = 0.2$ was used with points from $t^* = 0.000005$ to $t^* = 15$. Also for this parameter value, there is a slope of one for $t^* > 1$. Around $t^* = 1$, there is again a clear transition to higher slopes but there is no extended domain over which the slope equals three. Instead, there is a gradual increase in slope with shorter times, which is in line with the fact that the slope should go to infinity as $t \to 0$. Arguably, this slope transition is smoother than the one found for $h_0^* = 0.02$ but it seems that the break in the slope should still be detectable. Finding this break would suffice to constrain possible parameter combinations, just as was done previously [2,5]. Interestingly, a small value for h_0^* implies that the Dupuit–Forchheimer assumptions are valid and thus the Boussinesq equation (see below), which underlines the consistency of previous and present analyses.

Although sudden drawdown has traditionally received much analytical attention as a classic and extreme case, the impulse input that it mimics is in reality probably better described by a more gentle rise and fall of the recharge. Fig. 4 shows the results for a cosine-shaped input with two aquifer parameterizations ($h_0^* = 0.02$ and $h_0^* = 0.2$). With $\lambda = 1$, the rise and fall of the impulse takes place over a period comparable to the

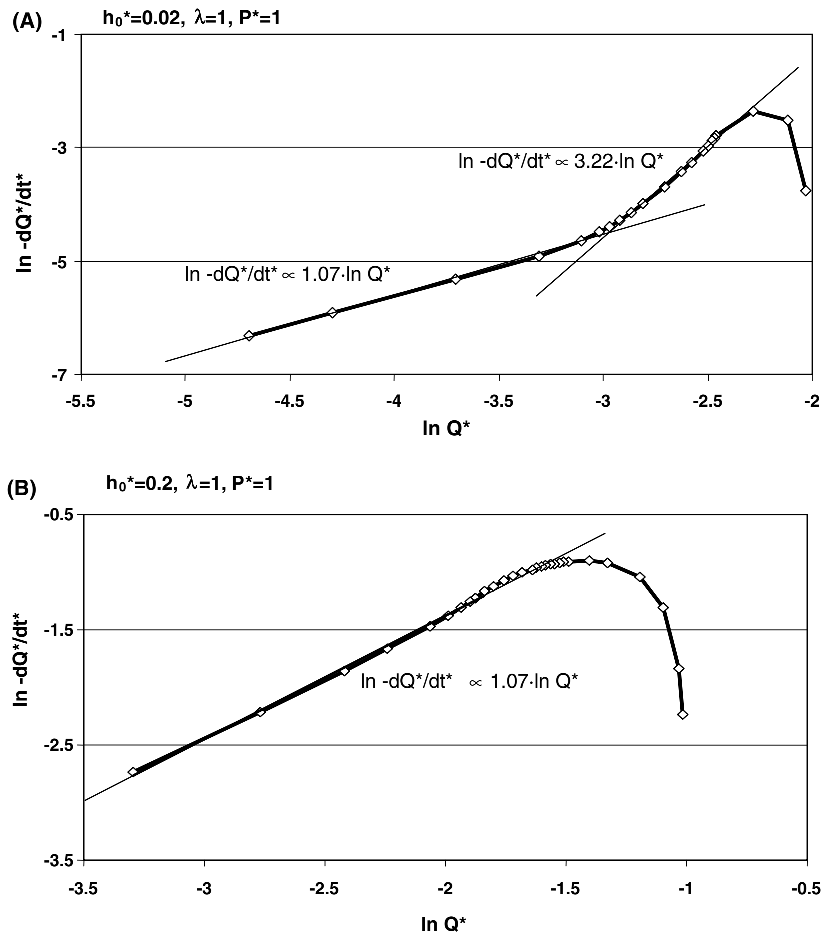

Fig. 4. (A) Recession after cosine-shaped pulse input with $h_0^* = 0.02$. (B) As (A) with $h_0^* = 0.2$. Formulas refer to thin regression lines that are fitted to (more or less) linear segments. The right side of the curves corresponds to short times, the left side corresponds to long times.

characteristic time of the aquifer. Still, for $h_0^* = 0.02$, the observed behavior exhibited by the $\ln(-dQ^*/dt^*)$ versus $\ln(Q^*)$ curve consists of a slope of one for long times and a slope of (around) three for medium to short times (Fig. 4A). For very short times, the slope becomes negative because the change in Q speeds up while Q diminishes when the recharge impulse starts its downward limb. The overall result is a clear transition which would enable a parameter estimation as suggested by Brutsaert and Lopez [2]. What is interesting is that the sharpness of the transition depends on the aquifer parameterization. As is shown in Fig. 4B, the transition between short- and long-time behavior becomes unobservable when, for the same impulse shape as in Fig. 4A, $h_0^* = 0.2$. This complicates the parameter estimation and shows that the proposed methods may work better for relatively shallow aquifers than for deeper aquifers.

3.2. Drawdown after equilibrium

Also for recession flow after an equilibrium has set in, the precise dynamics of the drawdown is important, as is the parameterization of the aquifer. Fig. 5A shows the $\ln(-dQ^*/dt^*)$ versus $\ln(Q^*)$ graph for a still relatively slow drawdown with $\lambda = 1$ in the order of the characteristic time of the system. Again, two domains can be distinguished that could be used to fix the parameters of the aquifer. The first domain covers large t, with $dQ^*/dt^* \propto Q^*$, which corresponds to the familiar slope of one in the graph. For each case examined here, this long-time behavior is valid, confirming the validity of a linear extinction coefficient in recession analysis. The second domain covers short t with $dQ^*/dt^* \propto Q^{*4.3}$. This second domain can clearly be distinguished from the first one although the transition is not as well defined as the one following an impulse input. One could argue that the transition is not from one linear domain to another but from a linear domain to a cosine-shaped line. If a slope is to be assigned to this domain, the indicated slope of 4.3 seems reasonable and definitely differs from the slope of three found after impulse inputs.

Although the transition in Fig. 5A may not be sharp, it can be distinguished and as such could support aquifer analysis. If the aquifer becomes deeper in comparison to

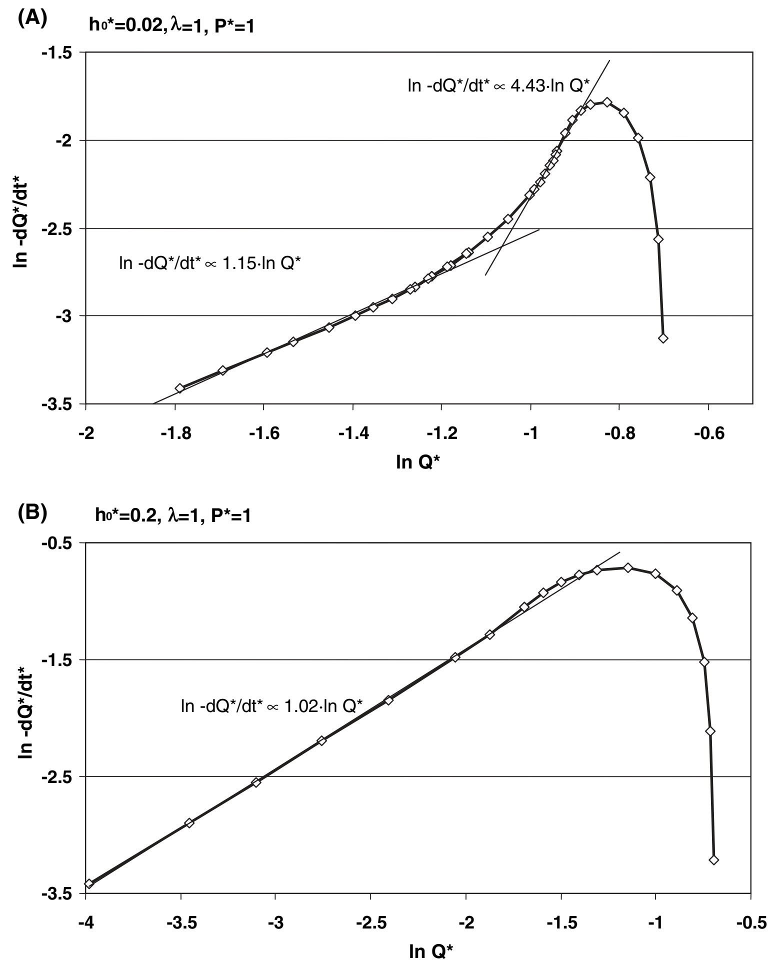

Fig. 5. (A) Recession after equilibrium $h_0^* = 0.02$. (B) As (A) with $h_0^* = 0.02$. Formulas refer to thin regression lines that are fitted to (more or less) linear segments. The right side of the curves corresponds to short times, the left side corresponds to long times.

its horizontal extent ($h_0/L = 0.2$ instead of $h_0/L = 0.02$) this is no longer possible, as is shown in Fig. 5B. Although the long-time behavior clearly corresponds to that of a linear reservoir, the transition to short time behavior is too smooth. This is similar to what happens after an impulse input. The speed of the transition, be it of an impulse, be it of the cessation of continuous rainfall, determines of course if a sufficiently sharp transition can be observed. Here, however, it becomes clear that what exactly constitutes a fast transition depends on the parameterization of the aquifer. If the aquifer is deep, a faster transition (in absolute terms) is needed than when the aquifer is shallow.

4. Comparison with Boussinesq equation

The approaches to aquifer characterization presented in the literature, were all based on solutions of the Boussinesq equation [2,8,6,5]. For the geometry under consideration here, there is a close structural similarity between the solutions for the two-dimensional Laplace equation and the one-dimensional Boussinesq equation. The solution of the Boussinesq equation for Q can be obtained by replacing the factors $\tanh(\frac{n\pi h_0}{L})$ with $\frac{n\pi h_0}{L}$ [9]. For sudden drawdown, for example, we obtain the following solution of the Boussinesq equation:

$$Q(t) = \sum_{n=1,3,\ldots}^{\infty} -K\frac{4\Delta h}{n\pi}\left(\frac{n\pi h_0}{L}\right)\cdot \exp\left(-\frac{K}{\mu}\left(\frac{n\pi h_0}{L}\right)\frac{n\pi}{L}t\right) \tag{21}$$

The validity of the Boussinesq equation hinges on whether the energy losses in the vertical direction can be neglected with respect to energy losses in the horizontal direction (Dupuit–Forchheimer or DF assumptions). For $x \ll 1$, we have $\tanh(x) = x + O(x^3)$ so as long as $\frac{n\pi h_0}{L} \ll 1$ we expect the Boussinesq equation to give results comparable to those based on the Laplace equation. For $t \to \infty$, the first terms dominate so as long as $h_0/L \ll 1$, Boussinesq will be valid. For $t \to 0$, however, higher terms also contribute and the Boussinesq results differ from the Laplace results. Physically, one could argue that after a sudden drawdown water will move significantly downward relatively fast, especially near the river, which is the area contributing most to $Q(t \to 0)$. The DF assumptions are, therefore, not valid for very short times for the given geometry and initial conditions. If we look at drawdown after equilibrium has set in, we know that initially, $Q_B = Q = P^*L$ so it may be expected that Boussinesq should give good results as long as $h_0/L \ll 1$.

These qualitative considerations are in general supported by the actual results so far. For very short times after sudden drawdown, the Laplace solution does not show the behavior predicted on the basis of Boussinesq ($\mathrm{d}Q^*/\mathrm{d}t^* \propto Q^{*3}$). For short to medium time, the aquifer does behave like Boussinesq ($\mathrm{d}Q^*/\mathrm{d}t^* \propto Q^{*3}$) provided $h_0/L \ll 1$. The same match between Laplace and Boussinesq holds for long times, if the linearization used here is valid. As remarked before, Boussinesq gives $\mathrm{d}Q^*/\mathrm{d}t^* \propto Q^{*3/2}$ for long times after complete drawdown and the present analysis does not give a solution based on Laplace for these boundary conditions [2,5]. The earlier predictions as reported in the literature, were, therefore, based on a different geometry and linearization. It would still be interesting to see what the Boussinesq solution looks like for the present geometry. One might argue that given the fact that Laplace describes actual aquifer behavior more correctly, it is only of academic interest to examine the Boussinesq solution. However, because Boussinesq is one-dimensional, the exact shape of the drain or river and its connection to the aquifer do not matter. To the Boussinesq equation, the fully penetrated aquifer (Fig. 1A) provides the same boundary condition as the partially penetrated aquifer (Fig. 1C). It is, therefore, interesting to look further into the Boussinesq solution because when the DF assumptions would hold, the complete analysis would be valid for a very broad set of river-aquifer connections.

Somewhat surprisingly, it turns out that in most cases, the Boussinesq equation predicts aquifer responses that are very similar to those found with the Laplace equation. Both in the case of a cosine-shaped impulse and in the case of a cosine-shaped cessation of recharge after equilibrium, the $\ln(-\mathrm{d}Q^*/\mathrm{d}t^*)$ versus

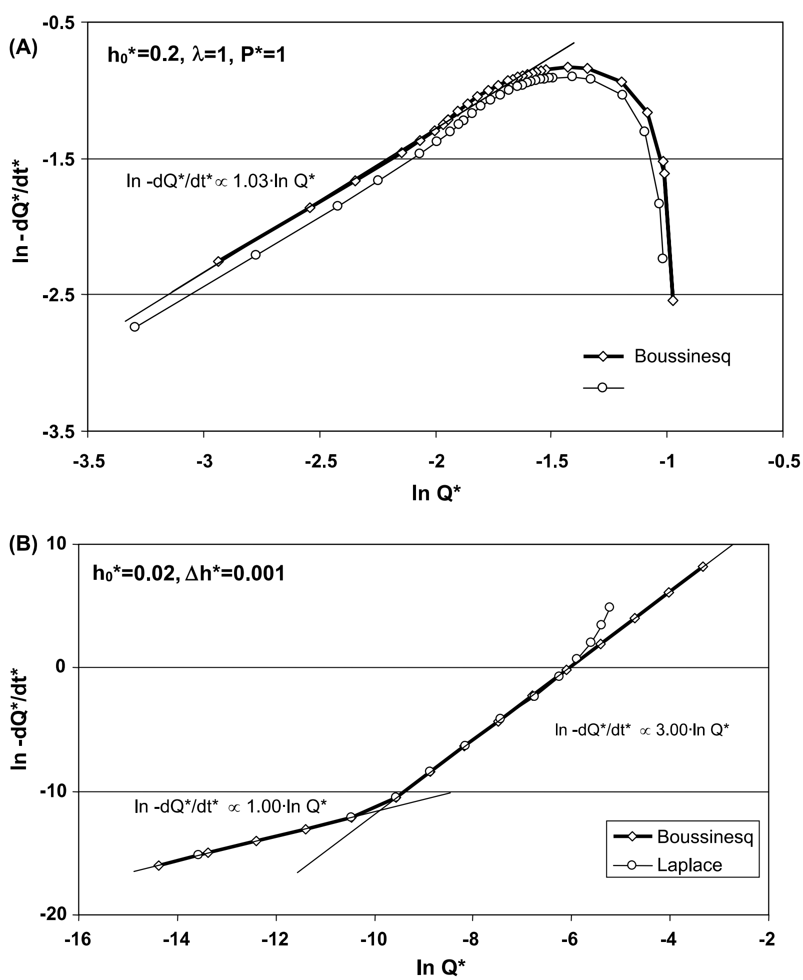

Fig. 6. Comparison of Boussinesq and Laplace solutions. (A) Impulse input. (B) Sudden drawdown. Formulas refer to thin regression lines that are fitted to (more or less) linear segments of the (thick) Boussinesq-based curves. The right side of the curves corresponds to short times, the left side corresponds to long times.

$\ln(Q^*)$ graphs have basically the same shape. This similarity holds for small values of $h_0^* = 0.02$, under which condition the DF assumptions would be valid because the vertical energy losses would be small compared to horizontal energy losses. More interestingly, the similarity in behavior is also present in the case when $h_0^* = 0.2$ when the DF assumptions are no longer really tenable. Fig. 6A shows Laplace and Boussinesq graphs for the case $h_0^* = 0.2$, following a cosine-shaped impulse. The remaining cases give equally close similarities but are not shown here. An important conclusion that may be drawn from these comparisons is that the exact boundary condition (fully or partially penetrated aquifer) does not seem to affect the curves that are of interest here. These analytical results concur with earlier findings by Szilagyi [7] that were based on numerical integration of the Laplace equation of a partially penetrated aquifer.

The last Boussinesq case that remains to be examined is the sudden drawdown. The case presented here resembles a drawdown as presented in Fig. 1A and can be linearized. As Fig. 6B shows, we finally find an extended straight slope of three in the $\ln(-\mathrm{d}Q^*/\mathrm{d}t^*)$ versus $\ln(Q^*)$ graph. The slope of three for short time, followed after an abrupt transition by a slope of one for long times, holds independent of the parameterization of h_0^*. This also becomes clear when one looks at Eq. (21) and realizes that

$$1/\sqrt{t} \approx \sum_{n=1,3,\ldots}^{\infty} 4 \cdot \exp(-n^2\pi \cdot t) \quad \text{for } 0 < t \ll 1 \qquad (22)$$

The equivalency in Eq. (22) holds up to five decimals for $t < 0.1$. The structure of this solution, $Q \propto t^{-1/2}$ for small t and $\ln(Q) \propto t$ for large t, is the same as that of the solution found by Poluborinova–Kochina for semi-infinite aquifers with zero depth boundary conditions in the drains (see for example [1]), which was later extended by Hogarth and Parlange [4] and Parlange et al. [5] to also cover finite aquifers. Although perhaps interesting from a theoretical point of view, this result is not very helpful from a practical point of view. As the earlier Laplace-based analysis shows, this Boussinesq-based result is not valid for (very) short times. This in turn may be ground to re-evaluate short time aquifer behavior in cases where the boundary conditions are better described by drains with zero-depth.

5. Conclusions

Full use was made of the very general character of the solution for the linearized Laplace equation [9]. Specifically, the effect of the dynamics of recharge and recharge cessation on the $\mathrm{d}Q/\mathrm{d}t - Q$ relation for short and long times, were examined. This was done because such short- and long-time aquifer behavior is at the basis of the aquifer parameterization method put forward in Brutsaert and Lopez [2] and Parlange et al. [5]. The analyzed geometry differs from that in previous analyses in that a significant water depth in the draining rivers is assumed. Four important conclusions can be drawn from the analysis presented here.

First, for the present geometry (Fig. 1A) and the Laplace solution, the $\ln(-\mathrm{d}Q^*/\mathrm{d}t^*)$ versus $\ln(Q^*)$ curve shows behavior similar to that as previously reported for zero-depth drainage (Fig. 1B) and the Boussinesq solution. The curve has a slope of one for long times ($t^* > 1$) and a slope of three for short times ($t^* < 1$). However, for very short times after sudden drawdown, the Laplace equation gives a slope that goes to infinity, showing that the Boussinesq equation is not valid for this rather theoretical situation. Also when the recharge impulse is more gentle, a three-to-one slope transition may be observed around $t^* = 1$.

Second, if recharge stops after the aquifer has come into equilibrium with a constant recharge rate, a transition between short- and long-time behavior can be observed as well. In the $\ln(-\mathrm{d}Q^*/\mathrm{d}t^*)$ versus $\ln(Q^*)$ curve, this transition is in general smoother than after impulse inputs and does for short times not have a slope of three. A slope of one for long times is observed, as it is indeed for each case examined here.

Third, there exists a relation between the parameterization of the aquifer and the sharpness of the transition between short and long times, which is of practical importance. Although a sharp transition between short and long time behavior can always be found if the change in recharge is sufficiently fast, it depends on the aquifer characteristics what exactly constitutes "sufficiently fast". For example, aquifers that are relatively deep, show smooth transitions or no clear transitions in reaction to recharge patterns that do give sharp transitions for shallower aquifers.

Fourth, the solution based on the Boussinesq equation gives the same general results as the solution based on the Laplace equation except for sudden drawdown. This is important because the one-dimensional Boussinesq equation cannot distinguish between fully and partially penetrated aquifers (Fig. 1A and C) making the general method basically independent of the exact river/aquifer interface. For the sudden drawdown case, it was shown that the solution of the Boussinesq equation for the present geometry (Fig. 1A) shows the same structural behavior for short times as previously found for the zero water depth geometry (Fig. 1B). Unfortunately, this solution is not valid for short times which may hinder parameter determination.

In general, it has been shown, for the geometry at hand, that recharge dynamics and aquifer parameterization determine to a large extent if a sharp transition between short- and long-term behavior can be observed. If a large scatter exists in the $\ln(-\mathrm{d}Q^*/\mathrm{d}t^*)$ versus $\ln(Q^*)$ graph for (very) short times, this may still be due to different shapes of the recharge recession. When such scatter is found, it may be possible to make at least a qualitative categorization of recharge events to improve estimation of aquifer parameters.

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