New approximate analytical technique to solve
Richards equation for arbitrary surface
boundary conditions

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Abstract. A general approximation for the solution to the one-dimensional Richards
equation is presented. It applies to arbitrary soil properties and boundary conditions but
only uniform initial conditions at current stage. The result is very accurate (within 2%) when
the diffusivity is constant, suggesting that the present general formulation is reliable,
since the approximation becomes increasingly accurate as the soil-water diffusivity
approaches a delta function.

Introduction

Parlange et al. [1992] extended the Heaslet and Alksne [1961] technique valid for power law diffusivities to solve the Bruce and Klute [1956] equation for arbitrary diffusivity. The present note uses the same idea to obtain solutions of Richards equation, that is, with gravity effects. Three main difficulties have to be overcome. First, Parlange et al. [1992] used the boundary condition of a constant water content at the surface, which is now replaced by an arbitrary condition, flux or water content, not necessarily constant in time. Second, Parlange et al. [1992] used an independently obtained expression for the sorptivity. Here we have to replace it by a general condition. Third, the effect of gravity has to be taken into account.

Theory

We consider

\[ \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( D \frac{\partial \theta}{\partial z} + \frac{\partial K}{\partial z} \right), \]

where \( \theta \) is the water content, \( D \) is the soil-water diffusivity, \( K \) is the soil-water conductivity, \( t \) is the time, and \( z \) is the vertical distance from the soil surface.

Since the initial water content is assumed to be uniform, equal to \( \theta_i \), using new variable \( (\theta - \theta_i) \) is equivalent to taking \( \theta_i = 0 \). At the surface the matric potential \( h_s \), or the water content \( \theta_s \), or the flux \( q \) are given as a function of time. With

\[ h_s = -S \frac{\partial \theta}{\partial z}, \]

where \( S \) is the soil-water conductivity, \( t \) is the time, and \( z \) is the vertical distance from the soil surface.

where better approximations for \( f(\tilde{\theta}, t) \) are sought. This approach is very useful for theoretical understanding of the structure of the solution as done by Sivapalan and Milly [1989], but its use to predict profiles has been very limited (for instance, see the example of Parlange and Braddock [1980]). On the other hand, the technique of Parlange et al. [1992] keeps the left-hand side of (3) simple and unchanged and replaces \( z \) by \( z + Mz^2 \) in the right-hand side of the equation. This approach leads to a simple and extremely accurate result in the case of Parlange et al. [1992].

Note that for mathematical convenience we use \( D(\theta) \) in (3); this applies as long as \( \partial h/\partial \theta \) is finite; otherwise we should use \( K \) and \( h \) instead of \( D \) and \( \theta \). In the following, we primarily consider the case \( \partial h/\partial \theta \) finite for simplicity, although this condition will be relaxed in subsequent applications of the method.

If we have a known relation between \( q \) and \( t \) for a given \( \theta_s(t) \), then only \( M(t) \) would remain unknown. Such a relation

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Paper number 96WR03846.
0043/1397/97/96WR-03846S09.00
is given by Barry et al. [1995, Equation (2)] for \( h_s \) constant. In this paper we shall provide a general alternative procedure to obtain both \( q(t) \) and \( M(t) \) (this being equivalent to fixing the sorptivity as in the work by Parlange et al. [1992]), although, in practice, the a priori knowledge of \( q(t) \) could simplify the analysis.

Following Parlange et al. [1992], the first condition is obtained by integration of (3), or

\[
\int_0^h \frac{\theta}{q} \frac{D}{\theta} d\theta = I + M(t) \int_0^h z^2 d\theta,
\]

where \( I \) is the cumulative infiltration,

\[
I = \int_0^t q(\theta) dI = \int_0^h z d\theta.
\]

Equation (5) requires a knowledge of \( f z^2 d\theta \). However, since the \( z^2 \) term is of lesser importance in (5), we can calculate this term with some inaccuracy, for example, from the value of \( z \) given in (3) without the \( z^2 \) term. First, we obtain by double integration of (1), the exact result

\[
\frac{d}{dt} \int_0^h z^2 d\theta = -2 \int_0^h [K \delta z/\delta \theta] d\theta + 2 \int_0^h D d\theta,
\]

and to estimate the right-hand side of that equation, we replace \( \delta z/\delta \theta \) using the z term only in (3), then (7) becomes

\[
\frac{d}{dt} \int_0^h z^2 d\theta = 2q \int_0^h \frac{\theta}{\theta} \frac{D}{\theta} d\theta - \int_0^h \frac{\theta}{q} \frac{D}{\theta} d\theta - K
\]

which yields \( f z^2 d\theta \) by integrating the right-hand side of (8) with time. An alternate approach when \( D \) increases rapidly with \( \theta \) is to recognize that \( f z^2 d\theta \) behaves like \( F^2/\theta_s \) and write

\[
\int_0^h \frac{\theta}{q} \frac{D}{\theta} d\theta = I^2 \int_0^h D d\theta/S'(-\theta),
\]

where \( S(\theta) \) is the sorptivity when \( \theta_s \) is constant. The result then becomes exact for short times, in general, and for all times as \( D \) approaches a delta function. This approximation of (9) follows the approach used earlier in the work of Parlange et al. [1985], but here we use it only to estimate the \( z^2 \) correction in (1). Undoubtedly, other alternatives to (8) or (9) could be developed that could be more convenient to use for a particular problem.

Finally, using (3) in (1) and imposing that it be satisfied in the limit of \( z \to 0 \) gives the condition

\[
2M = \frac{q}{\theta_s D_s} - \frac{d}{dt} \frac{1}{q} - \frac{1}{K_s},
\]

where \( D_s = D(\theta_s) \). Again, for particular examples it might be more convenient to use an alternate condition to (10), for example, as done by Parlange et al. [1992] where condition at the wetting front rather than the surface is utilized. It is also likely that (10) requires that \( \theta_s(t) \) and \( q(t) \) cannot have erratic behavior, and in particular does not take hysteresis into account.

The present solution having arbitrary boundary conditions is applicable to describe many phenomena. Several general applications will be discussed in other papers. In the following, the method is illustrated for cases when exact analytical solutions exist.

**Examples of a “Linear” and a “Near-Linear” Soil**

In this note we shall first consider the accuracy of the approximation in the simple case when (1) is linear. This is not to say that a “linear” soil is physically meaningful, but rather that since the expansion of (3) improves as \( D \) increases faster with water content, the linear soil provides a critical case to estimate the error of the expansion since it should be least accurate in that case. Sivapalan and Milly [1989], for instance, used this method very effectively to obtain an estimate of the maximum error caused by the time condensation approximation. We take as an example a linear soil with constant water content \( \theta_s \) at the surface and \( D = D_s \); \( K = K_s \theta_s/\theta_s \). We can always take

\[
D = 1; \quad K = \theta; \quad \theta_s = 1
\]

without loss of generality if we use \( \theta/\theta_s \) as reduced water content \( t \ K_s^2/D_s \) and \( z \ K_s/D_s \) as new dimensionless variables. Equation (5) gives at once, using (8) and (10),

\[
\frac{1}{q - 1} = I + q \int_0^t \frac{q}{q - 1} d\bar{t},
\]

which can be rewritten, in the simpler form,

\[
I = q \frac{2}{\sqrt{2q} - 1} \ln \left[ \frac{2q - 1 + 1}{2q - 1 - 1} \right],
\]

which yields \( I(t) \) using Runge-Kutta integration. Note that as \( t \to 0 \),

\[
I = 2 \sqrt{t/3} + 7t/15 + O(t^{3/2}).
\]

Using (9) instead of (8), (5) yields

\[
1/q - 1 = I + 3q t^{1/4},
\]

which can be integrated analytically, or,

\[
t = \frac{7}{8} \left[ I - \ln \left( 1 + I \right) \right] - \frac{3}{16} \frac{I^2}{I} + \frac{8}{24}
\]

\[
- \frac{1}{24} \left[ 64 + 24I + 9I^2 \right]^{1/2} + \frac{1}{16} I \left[ 64 + 24I + 9I^2 \right]^{1/2}
\]

\[
+ \frac{7}{8} \ln \left[ (64 + 24I + 9I^2)^{1/2} + 3I + 4 \right] - \frac{7}{8} \ln 12
\]

\[
+ \frac{7}{8} \arcsinh \left[ \frac{1}{72} \left( 104 + 6I \right) \sqrt{3} \right] - \frac{7}{8} \arcsinh \left[ \frac{\sqrt{3}}{9} \right].
\]

Note that as \( t \to 0 \),

\[
I = 2 \sqrt{t/3} + t/2 + O(t^{3/2}).
\]

The exact solution to the linear problem is of course well known and is given by

\[
I = 1 + t - (1 + t/2) \text{erfc} \left[ \sqrt{t/2} \right] + \sqrt{t/\pi} \exp \left[ -t/4 \right].
\]
and in particular, as $t \to 0$,

$$I = 2 \sqrt{t/\pi} + t/2 + O(t^{3/2})$$

Finally, the earlier solution for infiltration, known to be excellent for $D$ increasing rapidly with $t$ [Parlange et al., 1985; Haverkamp et al., 1990; Barry et al., 1995], becomes with present notations,

$$I = 2/[3(q - 1)]$$

or by integration,

$$t = I - 2 \frac{3}{3} \ln \left[ 1 + \frac{3I}{2} \right]$$

and in particular, for short times,

$$I = 2 \sqrt{t/3} + 2t/3 + O(t^{3/2}).$$

Figure 1 shows the four solutions; the two approximations in (13) and (16) have about a maximum error of 2% compared to the exact result of (18), with (16) slightly better. This is reflected also in the short-time expansions. The 2% positive error of (16) results from the error in the first term of (17), 3 replacing $\pi$, the second term, 1/2, being the exact result. Equation (13) is less accurate, but there is a compensation in errors; there too, 3 replaces $\pi$ in the first term, but this is overcompensated by the second coefficient 7/15 being significantly less than 1/2 so that $I$ has a 2% negative error. Finally, the earlier solution has a larger error as expected, reflected by 2/3 instead of 1/2 in the second term of (22).

Another interesting solution is for Burger’s equation [Clothier et al., 1981]. There we take an imposed flux at the surface with

$$D = 1; \quad K = \theta^0; \quad q = 1,$$

where the $K$ term makes (1) “minimally nonlinear” [Clothier et al., 1981], but it is still another example with an exact analytical solution. In particular, the water content at the surface increases with $t$ as [Clothier et al., 1981]

$$\theta_s = \text{erf} \sqrt{t}.$$
which can be combined with (27) to yield finally

\[
\frac{q}{\theta_s} t - z = \int_0^t \frac{D d \bar{\theta}}{\bar{\theta}} - K
\]

which has a traveling wave form and is the exact solution given by Ross and Parlange [1994, Eq. (1)]. If \(q/\theta_s\) happens to be equal to \(K(\theta_{sat})/\theta_{sat}\), where \(\theta_{sat}\) is the saturated water content, then as \(t \to \infty\), \(\theta \to \theta_{sat}\), and \(q \to K(\theta_{sat})\), and the predicted profile approaches the exact profile at infinity as expected.

When water content is imposed at the soil surface, Salvucci [1996] also obtained a convenient solution interpolating between short- and long-time behavior. His time expansion has also obtained a convenient solution interpolating between short- and long-time behavior. His time expansion has

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(Received August 22, 1996; revised November 18, 1996; accepted December 11, 1996.)