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Analytical approximation to the solutions of Richards' equation with applications to infiltration, ponding, and time compression approximation

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Abstract

A recent approach to solve Richards' equation is further improved. This approach brings understanding into the physical processes of infiltration and ponding. In particular we apply it to analyze the standard hydrologic tool of Time Compression Approximation (TCA). We also suggest that the new approach provides a more reliable alternative to TCA, e.g. for predicting ponding time. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Understanding water infiltration, ponding and drainage in soil relies on solving Richards' equation, or, in one dimension, finding solutions to,

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial z}\left[D\frac{\partial\theta}{\partial z}\right]-\frac{\partial K}{\partial z}, \tag{1}$$

where θ is the water content. In the following we shall assume for simplicity that the initial value of θ is constant with depth. Thus, we measure θ as the moisture content minus this initial value. D and K are the soil–water diffusivity and hydraulic conductivity, respectively, t is time and z the vertical distance from the soil surface. We are first going to review briefly a general approach to solve Eq. (1).

We represent the solution to Eq. (1) as an expansion in z, [8]

$$\int_{\theta}^{\theta_s}\frac{D\,\mathrm{d}\bar{\theta}}{q(\bar{\theta}/\theta_s)-K}=z+M(t)z^2+\cdots, \tag{2}$$

with the bar denoting the variable of integration. Eq. (2) is an extension of the formulations of Heaslet and Alksne [5] and Parlange et al. [10]. Although higher order expansions are possible, we obtain sufficient accuracy keeping only the terms in z and z^2, as shown [8]. Actually, keeping only the z-term is often accurate enough [7] and the z^2-term provides a further improvement. Here, q is the surface flux, θ_s the surface water content and $M(t)$ is an unknown function of time only. Note that Eq. (2) should be used only if no region of the soil is saturated. If saturation $\theta=\theta_{sat}$, is obtained for $z<z_{sat}$ (z_{sat} is the location of the saturated front) then for $z>z_{sat}$, Eq. (2) should be replaced by

$$\int_{\theta}^{\theta_{sat}}\frac{D\,\mathrm{d}\bar{\theta}}{q(\bar{\theta}/\theta_{sat})-K}=(z-z_{sat})+M(t)(z-z_{sat})^2+\cdots, \tag{3}$$

with q and z_{sat} related by

$$q=K_{sat}(h_s-h_{str})/z_{sat}, \tag{4}$$

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where $h_s - h_{str} > 0$ represents the matric potential measured at the surface above the water entry value.

In general, a boundary condition at the surface will be provided, e.g., $q(t)$ or $\theta_s(t)$ might be measured (subscript s refers to the surface). If one is given, then the other as well as $M(t)$ must be obtained by applying Eq. (2).

Integration of Eq. (2) by parts yields

$$\int_0^{\theta_s} \frac{D\bar{\theta}\,\mathrm{d}\bar{\theta}}{q(\bar{\theta}/\theta_s) - K} = I + M\int_0^{\theta_s} z^2\,\mathrm{d}\bar{\theta} + \ldots, \tag{5}$$

where I is the cumulative infiltration,

$$I = \int_0^{\theta_s} z\,\mathrm{d}\theta = \int_0^t q\,\mathrm{d}\bar{t}. \tag{6}$$

For the present method to apply the z^2-term in the expansion should be small compared to the z-term, thus we will estimate $M\int_0^{\theta_s} z^2\,\mathrm{d}\theta$ in Eq. (5) with some approximations. This estimate can be then used to yield $\theta_s(t)$ if q is known, for instance.

There are undoubtedly many ways to estimate $M \times \int_0^{\theta_s} z^2\,\mathrm{d}\theta$ and, in the following, two possible examples are given, which are applicable if D increases rapidly with θ, as is the usual case in practice for soil–water flow. In that case $\int z^2\,\mathrm{d}\theta$ case behaves like I^2/θ_s, (which is exact when D is a delta function) and we can take

$$\int_0^{\theta_s} z^2\,\mathrm{d}\theta \simeq 2I^2\int_0^{\theta_s} D\,\mathrm{d}\theta/S^2(\theta_s), \tag{7}$$

where $S(\theta_s)$ is the sorptivity for θ_s constant. The result is exact for short times and also if D were a delta function. Note that, if M is known, then Eqs. (5) and (7) can be solved for I. If we impose that Eq. (2) satisfies Eq. (1) as $z \to 0$, we obtain

$$2M = \frac{q}{\theta_s D_s} - \frac{\mathrm{d}\theta_s}{\mathrm{d}t}\frac{1}{q - K_s}, \tag{8a}$$

where $D_s = D(\theta_s)$. Note that M will normally be small, i.e., the two terms in the RHS of Eq. (8a) may cancel each other. Eqs. (5), (7) and (8a) give I for arbitrary soil properties.

M would be strictly zero if,

$$q = \frac{\theta_s D_s}{q - K_s}\frac{\mathrm{d}\theta_s}{\mathrm{d}t} \tag{9}$$

or, by integration,

$$I \simeq \int_0^{\theta_s} \frac{\theta D\,\mathrm{d}\theta}{q - K}, \tag{10}$$

this can be compared with the estimate of I obtained putting $M(t) = 0$ in Eq. (2), or

$$I \simeq \int_0^{\theta_s} \frac{\theta D\,\mathrm{d}\theta}{q(\theta/\theta_s) - K}. \tag{11}$$

Indeed if D increases rapidly with θ, then most of the contribution to the integrals occurs for $\theta \sim \theta_s$ and the two estimates are close to each other. In fact, they are identical, and then $M = 0$ exactly for the particular case when q and θ_s are proportional, i.e.

$$q = A\theta_s, \tag{12}$$

where A is a constant, whereas q and θ_s are functions of time. Thus, the present approach yields an interesting exact solution

$$z = \int_0^{\theta_s} \frac{D\,\mathrm{d}\bar{\theta}}{A\theta - K}, \tag{13}$$

when Eq. (12) holds at the surface. This exact solution can be used to obtain some insight into infiltration, drainage and ponding and will be discussed later. It can also be used to check the accuracy of numerical schemes [12].

Going back to Eq. (8a), some difficulties could arise in its application if $D(\theta_s)$ is poorly defined, which will often be the case especially near saturation. It is possible, however, to use Eq. (8a) to define an "effective" $D(\theta_s)$. To do so, we solve the problem of absorption (no gravity) with θ_s constant. Then Eqs. (5), (7) and (8a) give, with

$$I = S\sqrt{t}, \quad q = \frac{1}{2}S/\sqrt{t}, \tag{14}$$

$$2\theta_s\int_0^{\theta_s} D\,\mathrm{d}\theta = S^2\left[1 + \frac{1}{2}\frac{\int_0^{\theta_s} D\,\mathrm{d}\theta}{\theta_s D_s}\right] \tag{15}$$

or

$$2\theta_s D_s = \int_0^{\theta_s} D\,\mathrm{d}\theta S^2 \bigg/ \left[2\theta_s\int_0^{\theta_s} D\,\mathrm{d}\theta - S^2\right]. \tag{16}$$

Many accurate expressions for the sorptivity are available [2,4,7,9,11]. Eq. (16) can then be used to express $\theta_s D_s$. On the other hand, if D is a well-behaved function, then Eq. (15) can also be used to estimate S^2, e.g., if we take [1]

$$D = D_s(\theta/\theta_s)^n, \tag{17}$$

Eq. (15) yields,

$$S^2 = 2D_s\theta_s^2/(n + 3/2), \tag{18a}$$

which follows the form suggested by Brutsaert [2] and Parlange [7] and agrees with all expressions for the first two orders for n large, i.e., for normal soils when $n > 4$ [1]. This, in turn, suggests that, in Eq. (16), the expression to be used for S^2, could be

Table 1
Exact values [10] and approximate values of $S/(D_s^{1/2}\theta_s)$ for various values of n

n	1	2	3	4	5	6	7	8	9	10
Exact	0.88749	0.75305	0.66516	0.60213	0.55412	0.51599	0.48477	0.45861	0.43626	0.41689
Eq. (18a)	0.89443	0.75593	0.66667	0.60302	0.55470	0.51640	0.48507	0.45883	0.43644	0.41703
Eq. (18b)	0.87947	0.75053	0.66402	0.60150	0.55373	0.51574	0.48460	0.45848	0.43616	0.41681
Average	0.88695	0.75323	0.66535	0.60226	0.55422	0.51607	0.48483	0.45865	0.43630	0.41692

$$S^2 = 2\theta_s^{1/2}\int_0^{\theta_s}\theta^{1/2}D\,d\theta. \tag{19}$$

In general, the influence of the second term on the RHS of Eq. (2) tends to be small. For instance, dropping the second term in the bracket of Eq. (15), $(n+3/2)$ is replaced by $(n+1)$ in Eq. (18a), and the error on the sorptivity for n as low as 4 (about the lowest value for a real soil) is only 5%. For a linear, i.e., unrealistic, soil with $n=0$, the second term is relatively large, and the present theory should be used with caution. However, even then Eq. (18a) yields $S \simeq 1.1547\sqrt{D_s\theta_s^2}$, the exact coefficient is $1.1284(=2/\sqrt{\pi})$, only a 3% error, but without the second term the error jumps to 13%.

Eq. (19) presents a unique relationship between sorption and basic soil hydraulic properties. For instance such analytic relationship could be used to assess the influence of macro pores (which may cause a decrease in n) on sorption (e.g. Katul et al. [6]). Given that Eq. (19) is more general than the widely used Green and Ampt model, it could also be used to revisit the sorption estimates of Clapp and Hornberger [3]. Instead of Eq. (19) they estimate S^2 by $2\theta_s\int D\,d\theta$ (which is consistent with the Green and Ampt model), so that for the power law diffusivity they use their results for S^2 should be multiplied by $(n+1)/(n+(3/2))$.

Not only could Eq. (8a) lead to difficulties if D_s is poorly defined but it is somewhat less reliable than an integral condition as well, as it involves an estimate of derivatives of the water content near the surface. Thus if the derivative has a significant error the profile might look too square, or not square enough. For instance if the derivative is too small, and the profile too square, then the derivative will be too large near the wetting front, and vice versa. Thus an attractive alternative is to impose that Eq. (2) satisfies Eq. (1) near the wetting front, i.e. as $\theta \to 0$ and average the result with the condition at the surface. However the wetting front is finite only if $D(0)=0$, otherwise if $D(0)\neq 0$ Eq. (2) shows that z behaves like ln θ as $\theta \to 0$ and hence is infinite so that a small amount of water diffuses ahead of the main profile [10]. To remove that small amount of water we can simply replace D by $(D-D(0))$ in Eq. (2) to obtain a finite front at $z_F(t)$. For diffusivities increasing rapidly with θ, the water ahead of the main profile contains a very small amount of water and is normally unimportant [10]. Then we obtain the condition

$$\frac{dz_F}{dt} = (1+2Mz_F)q \tag{8b}$$

instead of Eq. (8a). For instance in the case of D given by Eqs. (17) and (8b)) yields, instead of Eq. (18a),

$$S^2 = D_s\theta_s^2\Big/\left[(n+1)\left(1-\frac{1}{2}\sqrt{n/(n+1)}\right)\right], \tag{18b}$$

which has essentially the same accuracy as Eq. (18a) for $n \geqslant 1$, but with errors of opposite signs as expected. For instance for $n=1$ Eqs. (18a) and (18b) yield sorptivities which differ from the exact value on the third significant figure, whereas the average differs on the fourth figure only. For $n \geqslant 6$ the errors drop by an order of magnitude, see Table 1. Even though Table 1 shows that the error of the estimates based on either Eq. (8a) or Eq. (8b) is always very small, it is clear that using the averages judiciously represents a significant improvement to the method, as expected. In the following not to complicate the algebra we shall use condition (8a) only, especially as it does not affect our conclusions.

2. Example of linear soils

Even though soils are not linear, this example provides a critical test of the approach since the results are more accurate the more rapidly D increases with water content. If we have $D=D_{sat}$; $K=K_{sat}\theta/\theta_{sat}$, we can always take

$$D_{sat}=1, \quad K_{sat}=1, \quad \theta_{sat}=1 \tag{20}$$

without loss of generality, if we use θ/θ_{sat} as reduced water content and tK_{sat}^2/D_{sat}, zK_{sat}/D_{sat} as new dimensionless variables.

Cumulative infiltration following saturation at the surface is given exactly by [8]

$$I = 1+t-\left(1+\frac{t}{2}\right)\operatorname{erfc}\left[\sqrt{t}/2\right]+\sqrt{t/\pi}\,\exp[-t/4], \tag{21}$$

whereas the present model gives, using only Eq. (8a) so we can apply directly an earlier result [8]

$$I + 3qI^2/4 = 1/(q-1). \tag{22}$$

In particular in the short time, gravity is unimportant, and up to terms of order t

$$I[21] \simeq 2\sqrt{t/\pi} + t/2 + \mathrm{O}(t^{3/2}) \quad (23)$$

and

$$I[22] \simeq 2\sqrt{t/3} + t/2 + (t^{3/2}). \quad (24)$$

In general Eqs. (21) and (22) differ by less than 3% [8], a remarkable result given that our approach is at its worst for a linear soil. This small error is due to the fact that Eq. (21) gives a sorptivity $S = 2/\sqrt{\pi}$ whereas Eq. (22) is consistent with $S = 2/\sqrt{3}$.

If, instead of imposing a constant θ at the surface, we impose a constant flux, q, θ at the surface is given exactly by [8],

$$\theta_s = q\left[1 - (1 + t/2)\,\mathrm{erfc}\frac{\sqrt{t}}{2} + \sqrt{t/\pi}\,\exp(-t/4)\right] \quad (25)$$

and the present model yields [8]

$$\frac{(\theta_s/q)^2}{1 - (\theta_s/q)} = t + \frac{3}{4}t^2\frac{q}{\theta_s}\left[\frac{q}{\theta_s} - \frac{1}{q}\frac{\mathrm{d}\theta_s/\mathrm{d}t}{1 - \theta_s/q}\right], \quad (26)$$

which is easily integrated. Again, concentrating on the short-time behavior, Eqs. (25) and (26) give, respectively,

$$\theta_s[25] = q[2\sqrt{t/\pi} - t/2 + \mathrm{O}(t^{3/2})] \quad (27)$$

and

$$\theta_s[26] = q[1.1105\sqrt{t} - 0.48875t + \mathrm{O}(t^{3/2})]. \quad (28)$$

If q is sufficiently large these equations can be used to predict the ponding time or,

$$t_p[25] = \pi/4q^2 + \pi^2/16q^3 + \mathrm{O}(1/q^4) \quad (29)$$

and

$$t_p[26] = 0.8109/q^2 + 0.6428/q^3 + \mathrm{O}(1/q^4) \quad (30)$$

which again differ by less than 3%.

We are now going to apply the results above to analyze the Time Compression Approximation (TCA) in predicting the ponding time, t_p, first for a constant flux.

3. TCA method

Being confident of the accuracy of the present method, it could be used as an alternative to other techniques, like TCA. It can also be used to estimate the error of other, less accurate, techniques, like TCA. As is well known [13] the prediction of t_p for a linear soil by the TCA in the absence of gravity is not very good (19% error), we can now explore the correction due to gravity. As illustrated in Fig. 1 call t'_p the prediction of the TCA method, which assumes that Eq. (21) or Eq. (23) for small gravity effect are relevant to a time τ, such that at that time the flux predicted by Eq. (23) be equal to the average rainfall rate,

$$q = 1/\sqrt{\pi\tau} + 1/2 + \ldots, \quad (31)$$

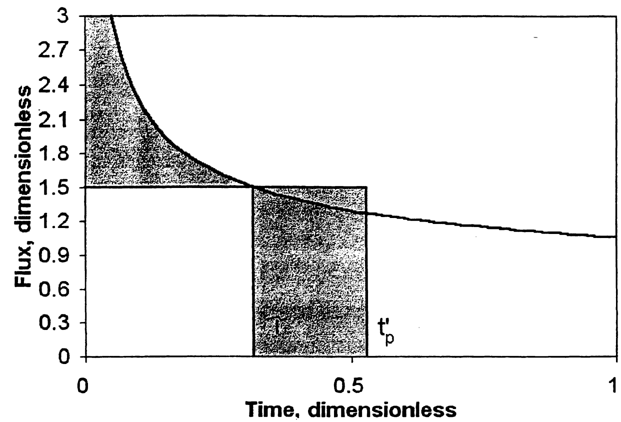

Fig. 1. Illustration of the time compression approximation when the average rainfall rate, q, is given. The infiltration rate for the case of ponding from time zero is equal to q at time τ, and the predicted ponding time t'_p, is such that the hatched areas above and below q are equal.

then at that time, I, given by Eq. (23) is equal to qt'_p,

$$qt'_p = 2/\sqrt{\tau/\pi} + \tau/2 + \cdots \quad (32)$$

Hence, eliminating τ,

$$t'_p = 2/\pi q^2 + 3/2\pi q^3 + \cdots \quad (33)$$

Comparing with Eq. (29) which yields the true value of t_p,

$$1 - t'_p/t_p = 1 - 8/\pi^2[1 - (\pi - 3)/q], \quad (34)$$

when $q \to \infty$, Eq. (34) of course shows an error of 19%, case without gravity. Gravity expressed by the term $8(\pi - 3)/q$ makes the error slightly worse. Obviously our new method is always much better than TCA, i.e., the error is always less than 3%. That is, we suggest that instead of using TCA it would be far more accurate to analyze the actual infiltration problem with the new analytical tool.

4. Application of exact solution to infiltration, ponding, and TCA

To pursue the analysis of TCA we are now considering a case when the rainfall rate is not constant, but is proportional to the surface water content θ_s, i.e., $q = A\theta_s$ where A is a constant, so that $M(t) = 0$, and the first term in the expansion of Eq. (2) yields the exact solution. We first neglect gravity and consider the diffusivity

$$D = D_s\theta. \quad (35)$$

Note that A has the dimension of conductivity, and since it is not a fixed soil property it would be artificial to take $A = 1$ as we did with K_{sat} in the previous section. Hence we also take $D_s \neq 1$ and all variables have their usual dimensions in the following. The solution with $\theta_s = 1$ at the surface needed to apply TCA is given by our approximation with $S^2 = 0.8D_s$ (see Eq. (18a)), or

$$I = \sqrt{0.8 D_s t}, \quad q = \sqrt{0.2 D_s / t}. \tag{36}$$

which is very accurate, i.e., less than 1/2%, see Table 1.

The exact solution, with $q = A\theta_s$ at the surface, is given by, see Eq. (5) for, $K = M = 0$,

$$I = \frac{1}{2} D_s \frac{q^2}{A^3} \tag{37}$$

or

$$I = (A^3/2D_s)t^2, \quad q = A^3 t / D_s. \tag{38}$$

In particular, the ponding time is given when $q = A$ or

$$A^2 t_p = D_s. \tag{39}$$

We can apply the TCA by equating the two fluxes given by Eqs. (36) and (38) to define a time τ. Then $I(\tau)$ from Eq. (36) is taken equal to $I(t'_p)$ from Eq. (38) to yield the approximate ponding time t'_p,

$$A^2 t'_p = 1.17 D_s, \tag{40}$$

a 17% error, comparable to the error of the previous case.

However, in this case we can also use TCA in a different way, where I is used as surrogate for time, as suggested by Smith et al. [14] and this method is illustrated in Fig. 2. Taking the $I(q)$, given by Eqs. (36) and (37) as equal we obtain a particular value of I. For this particular value of I, Eq. (38) yields, t'_p, or,

$$A^2 t'_p = 0.928 D_s, \tag{41}$$

which has only half the error of the estimate in Eq. (40), and is of course a preferable way to apply the TCA. But again our new approach, based on Eq. (5), is preferable to this improved TCA (for this particular case it yields the exact result).

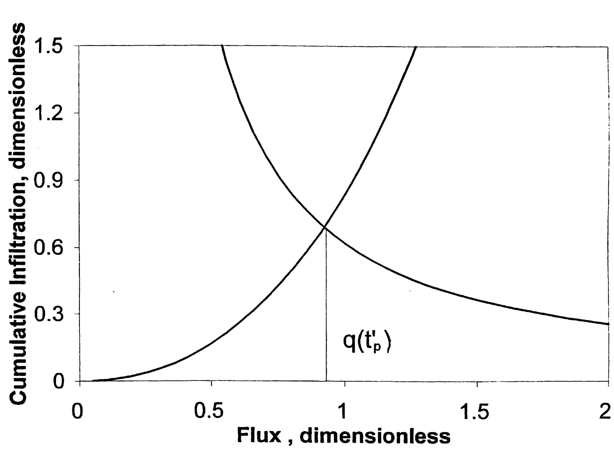

Fig. 2. Illustration of the time compression approximation when I is used as a surrogate for time. The value of q at ponding is obtained by the intersection of the two $I(q)$ curves. The one increasing with q represents the actual rainfall event and the decreasing one represents the case of ponding from time zero. Knowing the rainfall rate as a function of time yields the approximate ponding time t'_p when the rainfall rate is equal to q.

Corrections due to gravity can also be considered following Smith et al. [14] prescription. For instance taking still $D = D_s\theta$ and $K = K_s\theta^2$, $q = A\theta_s$, the exact solution yields,

$$At - z = \frac{D_s}{K_s} \ln \frac{A}{A - K_s\theta} \tag{42}$$

and, for $\theta_{sat} = 1$,

$$t_p = \frac{D_s}{K_s A} \ln \frac{A}{A - K_s} \tag{43}$$

and for $A \gg K_s$, i.e., for small gravity corrections,

$$t_p \simeq \frac{D_s}{A^2} + \frac{D_s K_s}{2A^3} + \cdots \tag{44}$$

Using the approximate theory for infiltration when $\theta_s = 1$, yields, for $q \gg 1$,

$$I = 0.4 \frac{D_s}{q} + \frac{2 D_s K_s}{9 q^2} + \cdots \tag{45}$$

Equating this $I(q)$ with the exact $I(q)$,

$$I = \frac{D_s q^2}{2A^3} + \frac{1}{3} \frac{D_s K_s q^3}{A^5} + \cdots \tag{46}$$

gives

$$q/A \simeq 0.928[1 - 0.0066\, K_s/A + \cdots], \tag{47}$$

which, used in the exact $q(t)$ relationship,

$$A^2 t = D_s q/A + D_s K_s q^2 / 2A^3 + \cdots \tag{48}$$

gives

$$A^2 t'_p = 0.928 D_s + 0.424 D_s K_s / A + \cdots \tag{49}$$

Interestingly, comparing this result to that of Eq. (44) shows that the gravity correction increases slightly the error of the estimate of ponding time. As in the case of the linear soil gravity increases the error of the TCA, because gravity delays the onset of ponding.

5. Conclusion

A new approximate method to solve Richards' equation seems to be general, simple and accurate. It may well prove to be the method of choice to estimate the accuracy of standard hydrologic tools like TCA. Thus this new method provides an additional hydrologic tool, which can either be used directly to analyze infiltration data or more generally to assess the reliability of other techniques by considering simple examples, as we have done here with TCA.

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