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Steepness expansion for free surface flows in coastal aquifers

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Abstract

Free surface flow of groundwater in aquifers has been studied since the early 1960s. Previous investigations have been based on the Boussinesq equation, derived from the non-linear kinematic boundary condition. In fact, the Boussinesq equation is the zeroth-order equation in the shallow-water expansion. A key assumption in this expansion is that the mean thickness of the aquifer is small compared with a reference length, normally taken to be the linear decay length. In this study, we re-examine the expansion scheme for free surface groundwater flows, and propose a new expansion wherein the shallow-water assumption is replaced by a steepness assumption. A comparison with experimental data shows that the new model provides a better prediction of water table levels than the conventional shallow-water expansion. The applicable ranges of the two expansions are exhibited.

Keywords: Shallow water expansion; Steepness expansion; Coastal aquifers; Groundwater flow

1. Introduction

There have been numerous investigations of groundwater head fluctuations in coastal aquifers where the forcing of oceanic tides is analysed. Most studies are based on the Boussinesq equation with the Dupuit assumption (Dagan, 1967; Parlange et al., 1984; Nielsen, 1990; Li et al., 2000). In fact, the Boussinesq equation is the zeroth-order term in the shallow-water expansion of the non-linear model describing the flow of fluid in an unconfined aquifer (Friedrichs, 1948; Barry et al., 1996). Recently a few higher-order perturbation solutions for free surface flow of groundwater were derived based on

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a shallow-water expansion (Nielsen et al., 1997; Teo et al., 2003). Among these Nielsen et al. (1997) attempted to relaxed the shallow aquifer assumption, based on Boussinesq equation. However, the Boussinesq equation is the linearised shallow-water equation. Thus, Nielsen's model will miss out some higher-order components originally generated from the non-linear shallow-water expansion. Furthermore, Nielsen et al. (1997) only considered the case of a vertical beach, not sloping beach. Teo et al. (2003) directly established the higher-order shallow-water expansion for finite amplitude watertable fluctuations. These analyses (Neilsen et al., 1997; Teo et al., 2003) demonstrate that the higher-order components have a significant influence on water table fluctuations.

In the shallow-water expansion, the shallow-water parameter ε is defined as the ratio of the mean thickness of the aquifer (D) to a reference length, normally the linear decay length (L) (Teo et al., 2003):

$$\varepsilon = \frac{D}{L} = \sqrt{\frac{n_e \omega D}{2K}} \text{ where } L = \sqrt{\frac{2KD}{n_e \omega}}, \tag{1}$$

where n_e is the soil porosity, K the hydraulic conductivity and ω the tidal frequency.

The shallow-water expansion is valid for $\varepsilon \ll 1$. This assumption will be invalid for conditions such as: high frequencies, deeper aquifers and low hydraulic conductivity. For example, with $D=5$ m, $T=0.5$ d (T is the tide period), $\omega=4\pi/d$ and $n_e=0.3$, K must satisfy $K>9.42$ m/day to even satisfy $\varepsilon<1$. Thus it would be useful to derive a new representation to cover wider ranges of the free surface flow of groundwater.

An alternative parameter that may used to replace the shallow-water parameter is the *wave steepness* parameter, defined by

$$\xi = \frac{A}{L}, \tag{2}$$

where A is the tidal wave amplitude, which is normally much smaller than L, i.e. $\varepsilon \ll 1$.

Here we derive a new analytical solution for tide-induced fluctuations in coastal aquifers, based on the wave steepness ξ. The new solution will provide a wider applicable range compared with the conventional shallow-water expansion.

2. Steepness expansion

2.1. Boundary value problem

In this study, the flow is assumed to be homogeneous and incompressible in a rigid porous medium. The flow configuration is shown in Fig. 1, where $h(x,t)$ is the dynamic water table height, D is the still water table height and β is the beach slope. The effects of seepage through the face are neglected in this study. Incompressibility and conservation of mass leads to Laplace's equation for the hydraulic potential head $\phi(x,z,t)$ (Bear, 1972):

$$\phi_{xx} + \phi_{zz} = 0, \tag{3}$$

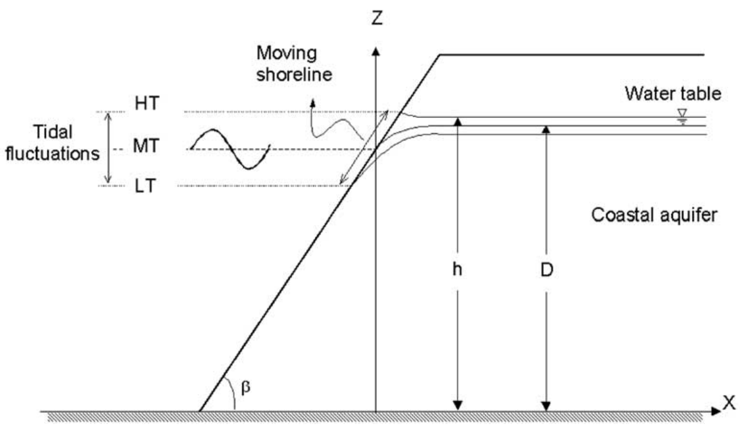

Fig. 1. Schematic diagram of tidal water table fluctuations in a coastal aquifer.

Eq. (3) will be solved subject to appropriate boundary conditions. First, the watertable heights at the boundary of the ocean and coast (i.e. $x=x_0(t)$) are equal to the specified tidal variation, i.e.

$$\phi(x_0(t), t) = D(1 + \alpha\cos(\omega t)), \quad x_0(t) = A\cot\beta\cos(\omega t), \tag{4a}$$

where $\alpha=A/D$ is the dimensionless amplitude parameter, A is the amplitude of tidal waves and D is the mean thickness of coastal aquifer.

Second, a zero-flux condition is applied at the impermeable base of the aquifer,

$$\phi_z = 0 \text{ at } z = 0. \tag{4b}$$

Third, the gage pressure at the free surface is assumed to be zero, which leads to

$$\phi = h \text{ on } z = h, \tag{4c}$$

together with the kinematic boundary condition

$$n_e\phi_t = K(\phi_x^2 + \phi_z^2) - K\phi_z \text{ on } z = h. \tag{4d}$$

Finally, the water table fluctuation will vanish far from the coast if there is no other recharge/input,

$$\phi_x \to 0 \text{ as } x \to \infty. \tag{4e}$$

2.2. Non-dimensional equations

It is convenient to introduce the following non-dimensional variables (Teo et al., 2003):

$$X = \frac{x}{L} - \xi\cot(\beta)\cos(\omega t),\ T = \omega t,\ Z = \frac{z}{D},\quad H = \frac{h}{D},\ \Phi = \frac{\phi}{D} \text{ and } \alpha = \frac{A}{D}. \tag{5}$$

The definitions (5) contain two non-dimensional parameters: the steepness (ξ) and the amplitude (α). The amplitude parameter α, representing the ratio of tidal amplitude (A) to the mean thickness of the aquifer (D), is normally less than unity. The applicable range of the steepness (ξ) has been discussed previously. Thus, there are three independent parameters defined by the material and boundary conditions: ξ, α and β. The approximate solution is constructed assuming that ξ and α are small, and allowing for a large range of β ($0<\beta<\pi/2$).

To apply the perturbation technique to the non-linear boundary value problem (3) and (4a–4e), the water table height (H) and potential head (Φ) are expressed in powers of the steepness parameter (ξ)

$$\Phi = \sum_{n=0}^{\infty}\xi^n\Phi_n \text{ and } H = \sum_{n=0}^{\infty}\xi^n H_n, \tag{6}$$

so that to zeroth and first-order

$$O(1): 2H_{0T} = (H_0H_{0X})_X, \tag{7a}$$

$$O(\xi): 2[H_{1T} + \sin(T)\cot(\beta)H_{0X}] = (H_0H_1)_{XX}, \tag{7b}$$

with boundary conditions

$$H_0(0,T) = 1 + \alpha\cos(T),\ H_1(0,T) = 0, \tag{8a}$$

$$H_{0X}(\infty,T) = H_{1X}(\infty,T) = 0. \tag{8b}$$

The detailed derivation of (3–8) is given in Appendix A. Note that we attempt to solve the water table heights ($h(x, t)$) from the above boundary value problem, not the potential function ($\phi(x,z,t)$) in the interior domain.

2.3. Zeroth-order approximation

Since (7a) is non-linear, we expand the solution in powers of α, defining H_0 as

$$H_0 = 1 + \sum_{n=1}^{\infty}\alpha^n H_{0n}. \tag{9}$$

From Eq. (7a), the equations to be solved for the zeroth-order approximation in ξ are:

$$O(\alpha)\ 2H_{01T} = H_{01XX}, \tag{10a}$$

$$O(\alpha^2)\ 2H_{02T} = H_{02XX} + (H_{01}H_{01X})_X \tag{10b}$$

$$O(\alpha^3)\ 2H_{03T} = H_{03XX} + (H_{01}H_{02})_{XX} \tag{10c}$$

with boundary conditions

$$H_{01}(0,T) = \cos(T), \text{ and } H_{01X}(\infty,T) = 0, \tag{11a}$$

$$H_{02}(0,T) = 0, \text{ and } H_{02X}(\infty,T) = 0, \tag{11b}$$

$$H_{03}(0,T) = 0, \text{ and } H_{03X}(\infty,T) = 0. \tag{11c}$$

The solution of the zeroth-order boundary value problem can be written as

$$H_{01} = e^{-X}\cos(\eta_1), \tag{12a}$$

$$H_{02} = \frac{1}{4}\left(1 - e^{-2X}\right) + \frac{1}{2}\left[e^{-\sqrt{2}X}\cos(\eta_2) - e^{-2X}\cos(2\eta_1)\right], \tag{12b}$$

$$\begin{aligned} H_{03} = {} & \frac{\sqrt{2}X}{8}\cos\left(\eta_1 + \frac{\pi}{4}\right) + \frac{1}{20}(\mathrm{e}^{-3X} - \mathrm{e}^{-X}) \\ & \times[11\cos(\eta_1) + 2\sin(\eta_1)] \\ & + \frac{3+2\sqrt{2}}{8\sqrt{2}}\left[\mathrm{e}^{-\sqrt{3}X}\cos(\eta_3) - \mathrm{e}^{-(\sqrt{2}+1)X}\cos(\eta_4)\right] \\ & + \frac{1}{4}\left[\mathrm{e}^{-X}\cos(\eta_1) - \mathrm{e}^{-(\sqrt{2}+1)X}\cos(\eta_5)\right] \\ & + \frac{1}{8\sqrt{2}}\left[\mathrm{e}^{-(\sqrt{2}+1)X}\sin(\eta_5) - \mathrm{e}^{-X}\sin(\eta_1)\right] \\ & + \frac{3}{8}\left[\mathrm{e}^{-\sqrt{3}X}\sin(3\eta_1) - \mathrm{e}^{-\sqrt{3}X}\sin(\eta_3)\right], \end{aligned} \tag{12c}$$

where

$$\begin{aligned} & \eta_1 = T - X,\ \eta_2 = 2T - \sqrt{2}X,\ \eta_3 = 3T - \sqrt{3}X, \\ & \eta_4 = 3T - \left(\sqrt{2}+1\right)X,\ \eta_5 = T - \left(\sqrt{2}-1\right)X. \end{aligned} \tag{13}$$

2.4. First-order approximation

We now consider Eq. (7b). Again, we expand the solution in powers of α, defining H_1 as

$$H_1 = \sum_{n=0}^{\infty} \alpha^n H_{1n} \tag{14}$$

From Eq. (7b), the equations to be solved for the first-order approximation in ξ are:

$$O(\xi\alpha^0) : 2H_{10T} = H_{10XX,} \tag{15a}$$

$$\begin{aligned} & O(\xi\alpha) : 2H_{11T} + 2\sin(T)\cot(\beta)H_{01X} \\ & = H_{11XX} + (H_{10}H_{01})_{XX}, \end{aligned} \tag{15b}$$

$$\begin{aligned} & O(\xi\alpha^2) : 2H_{12T} + 2\sin(T)\cot(\beta)H_{02X} \\ & = H_{12XX} + (H_{01}H_{11} + H_{01}H_{02})_{XX}. \end{aligned} \tag{15c}$$

with boundary conditions

$$H_{10}(0,T) = 0, \text{ and } H_{10X}(\infty,T) = 0, \tag{16a}$$

$$H_{11}(0,T) = 0, \text{ and } H_{11X}(\infty,T) = 0, \tag{16b}$$

$$H_{12}(0,T) = 0, \text{ and } H_{12X}(\infty,T) = 0. \tag{16c}$$

The solution of the first-order boundary value problem can be written as

$$H_{10} = 0, \tag{17a}$$

$$\begin{aligned} H_{11} = {} & \frac{1}{\sqrt{2}}\cot(\beta)\left[\frac{1}{\sqrt{2}} - \mathrm{e}^{-X}\cos\left(X - \frac{\pi}{4}\right)\right. \\ & + e^{-\sqrt{2}X}\cos\left(\eta_2 + \frac{\pi}{4}\right) \\ & \left. - \mathrm{e}^{-X}\cos\left(\eta_1 + T + \frac{\pi}{4}\right)\right], \end{aligned} \tag{17b}$$

$$\begin{aligned} H_{12} = {} & \cot(\beta)\left\{\frac{1}{2}\left[\mathrm{e}^{-2X}\cos(T) - \mathrm{e}^{-X}\cos(\eta_1)\right]\right. \\ & + \frac{1}{2}\left[\mathrm{e}^{-\sqrt{3}X}\cos\left(\eta_3 + \frac{\pi}{4}\right)\right. \\ & \left. - \mathrm{e}^{-\sqrt{2}X}\cos\left(T + \eta_2 + \frac{\pi}{4}\right)\right] \\ & + \frac{1}{2}\left[\mathrm{e}^{-X}\cos\left(\eta_1 + \frac{\pi}{4}\right)\right. \\ & \left. - \mathrm{e}^{-\sqrt{2}X}\cos\left(\eta_2 - T + \frac{\pi}{4}\right)\right] \\ & + \frac{1}{\sqrt{2}}\left[\mathrm{e}^{-2X}\cos\left(\eta_1 - X + \frac{\pi}{4}\right)\right. \\ & \left. - \mathrm{e}^{-X}\cos\left(\eta_1 + \frac{\pi}{4}\right)\right] \\ & + \frac{1}{\sqrt{2}}\left[\mathrm{e}^{-\sqrt{3}X}\cos\left(\eta_3 + \frac{\pi}{4}\right)\right. \\ & \left. - \mathrm{e}^{-2X}\cos\left(3\eta_1 + X + \frac{\pi}{4}\right)\right] \\ & + \frac{3+2\sqrt{2}}{8}\left[\mathrm{e}^{-\sqrt{3}X}\cos\left(\eta_3 + \frac{\pi}{4}\right)\right. \\ & \left. - \mathrm{e}^{-(\sqrt{2}+1)X}\cos\left(\eta_4 + \frac{\pi}{4}\right)\right] \\ & + \frac{1}{2\sqrt{2}}\left[\mathrm{e}^{-X}\cos\left(\eta_1 + \frac{\pi}{4}\right)\right. \\ & \left. - \mathrm{e}^{-(\sqrt{2}+1)X}\cos\left(\eta_5 + \frac{\pi}{4}\right)\right] \\ & + \frac{1}{4\sqrt{2}}\left[\mathrm{e}^{-(\sqrt{2}+1)X}\sin\left(\eta_5 + \frac{\pi}{4}\right)\right. \\ & \left.\left. - \mathrm{e}^{-X}\cos\left(\eta_1 + \frac{\pi}{4}\right)\right]\right. . \end{aligned} \tag{17c}$$

Note that, from (17b and 17c), H_{11} and H_{12} will vanish for the special case of a vertical beach (i.e. $\beta = \pi/2$). As reported in Teo et al. (2003), the solutions for vertical beach only appears in the even-order terms.

3. Results and discussion

The major difference between previous shallow-water expansions and the new steepness expansion is the use of a different perturbation parameter, i.e. ξ instead of ε. Fig. 2 shows an estimate of the applicable ranges corresponding to the two assumptions, $\varepsilon \ll 1$ and $\xi \ll 1$, as a function of K. In general, a perturbation parameter should be less than 0.5

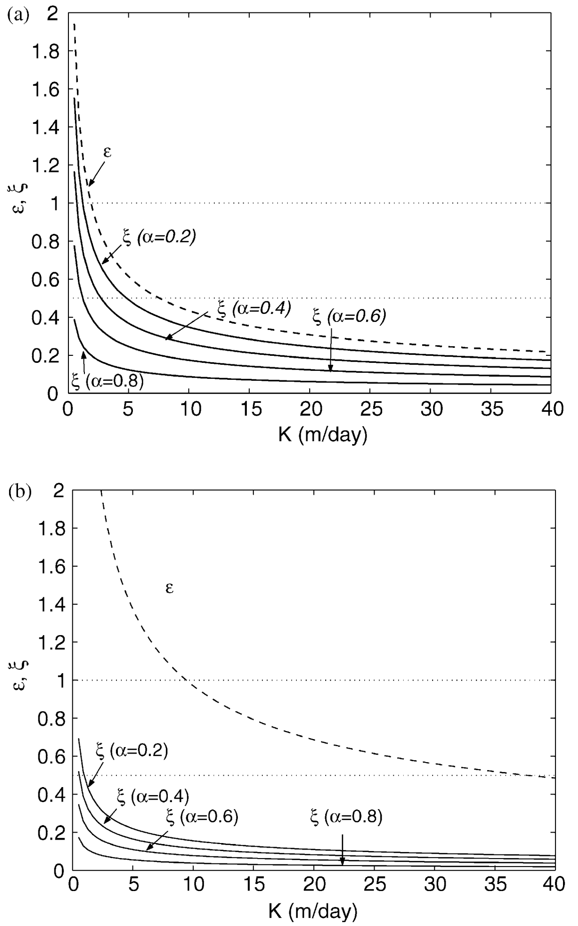

Fig. 2. Distribution of the shallow-water parameter (ε) and steepness parameter (ξ) versus hydraulic conductivity (K) (a) D=1 m and D=5 m.

Table 1
Properties of tidal waves and coastal aquifers in experimental work

Properties	Vertical beach	Sloping beach
Hydraulic conductivity (m/d)	40.6	114
Aquifer thickness (m)	1.094	1.01
Wave period (s)	772	348
Amplitude (m)	0.235	0.204
Shallow-water parameter (ε)	0.752	0.469
Amplitude parameter (α)	0.214	0.202
Steepness parameter (ξ)	0.161	0.095
Slope (β)	$\pi/2$	0.202 (rad)

(at least), see Kevorkian and Cole (1981), and when the parameter nears unity the expansion will be unreliable. As shown in Fig. 2, if we choose $\varepsilon = 0.5$ as the critical value, the shallow-water expansion will

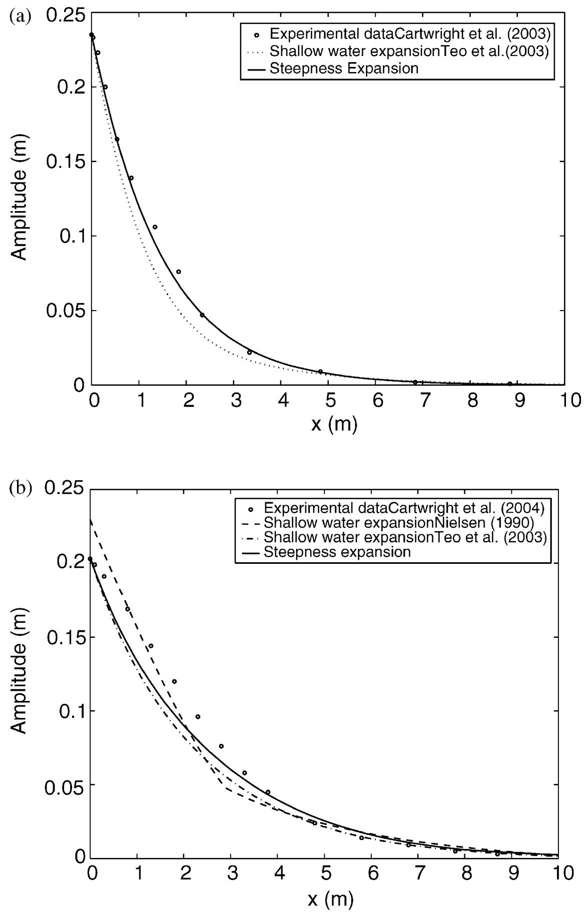

Fig. 3. Comparison of the two models against experimental data in (a) a vertical beach and (b) a sloping beach.

be invalid when $K < 8$ m/day with $D = 1$ m and $K < 37$ m/day with $D = 5$ m. Clearly the steepness parameter provides a much wider range of applicability for a range of the amplitude parameter, $\alpha = A/D$. Thus using the steepness parameter to replace the shallow-water parameter is a more versatile option.

To verify the theoretical models, experimental data from Cartwright et al. (2003, 2004), the shallow-water expansion (Teo et al., 2003) and the present steepness expansion are compared for a vertical beach and a sloping beach with $\beta = 0.202$ rad. The input data used is shown in Table 1 while the results for a vertical beach are illustrated in Fig. 3(a). The steepness expansion provides a better prediction of the water table level than the shallow-water expansion. Referring to Table 1, the shallow-water parameter $\varepsilon = 0.752$, which is close to one, while the steepness parameter $\xi = 0.161$, much less than one. Thus, for this example, the steepness parameter is a more suitable perturbation parameter than the shallow-water parameter.

The second comparison is for a sloping beach with $\beta = 0.202$. It includes a comparison with the results of Nielsen (1990) and is illustrated in Fig. 3(b). In this case the steepness expansion provides a slightly better prediction than the shallow expansion, while Nielsen's solution does not fully satisfy the boundary condition at $X = 0$ (Li et al., 2000; Teo et al., 2003). It is also noted that there is a disagreement between all theoretical solutions and the experiments near

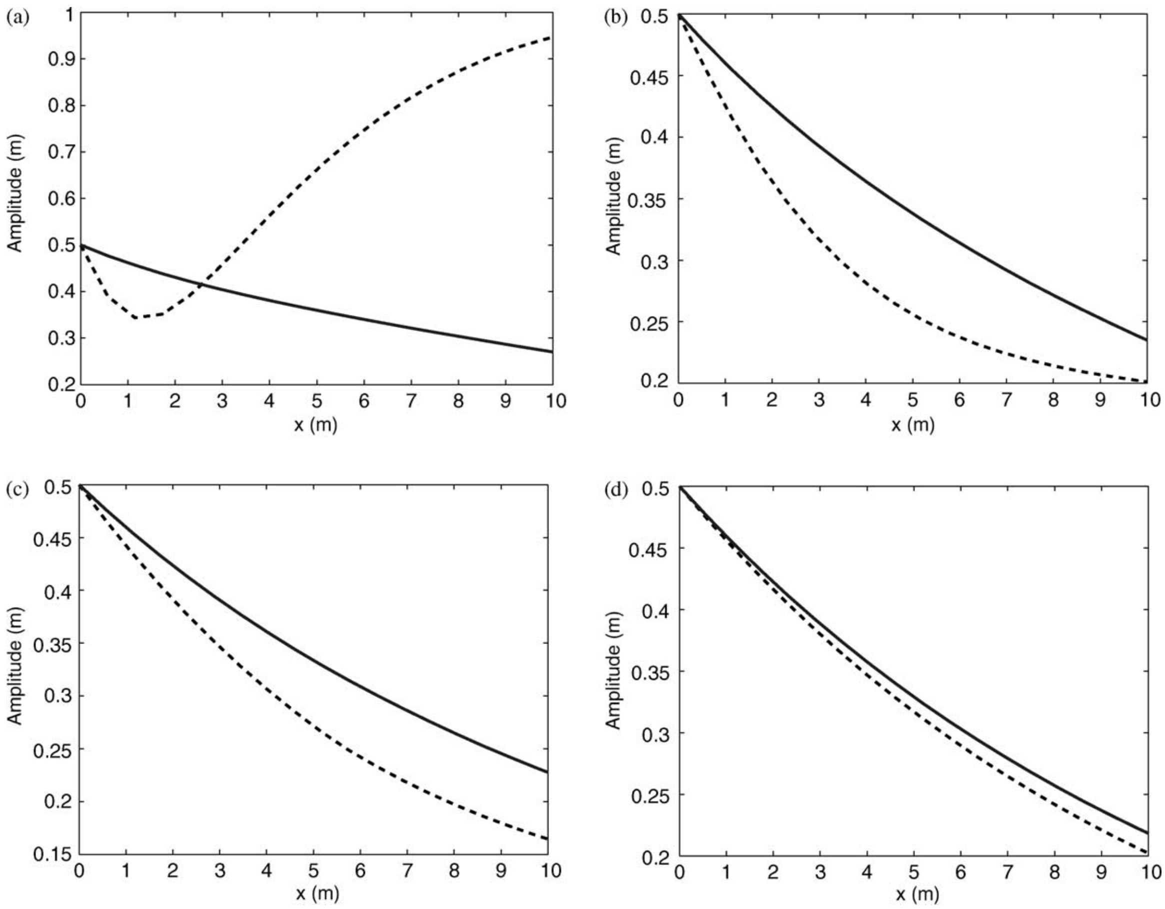

Fig. 4. Comparison of the two models for various values of K. Solid lines = steepness expansion, dashed lines = shallow-water expansion. (a) $K = 1$ m/d, (b) $K = 5$ m/d, (c) $K = 10$ m/d, and (d) $K = 50$ m/d.

the ocean wave/inland interface. This may come from the neglect of seepage at the beach face in the models.

Fig. 4 illustrates the amplitudes of the tide-induced water table fluctuations versus horizontal distance for both expansions. The two expansions give similar results for the case of high hydraulic conductivity (for example $K=5$ m/day). The difference between the two expansions increases as K decreases. When K is smaller than a particular value, for example $K=1$ m/d in Fig. 4, the shallow-water expansion is clearly incorrect. The expansion diverges as X increases because the parameter is greater than one.

4. Conclusion

In this study the traditional assumption of the shallow-water expansion is removed and a new perturbation parameter, the steepness (ξ), is introduced. Specific examples and comparison with experimental data demonstrate that the steepness expansion provides a better prediction of tide-induced water table fluctuations in coastal aquifer than the conventional shallow-water expansion. Additionally, the shallow-water expansion cannot be used for the case of low hydraulic conductivity, while the steepness expansion is applicable.

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Appendix A. Derivations of non-dimensional boundary value problem

In this appendix, the detailed derivation of Eq. (3–8) will be given.

Based on the non-dimensional transformation, (5), we have

$$\frac{\partial\phi}{\partial x} = \frac{D}{L}\left(\frac{\partial\Phi}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial\Phi}{\partial T}\frac{\partial T}{\partial x}\right) = \frac{\xi}{\alpha}\frac{\partial\Phi}{\partial X}, \tag{A1}$$

$$\frac{\partial\phi}{\partial t} = D\left(\frac{\partial\Phi}{\partial X}\frac{\partial X}{\partial t} + \frac{\partial\Phi}{\partial T}\frac{\partial T}{\partial t}\right) = \omega D\left(\xi\cot(\beta)\sin(T)\frac{\partial\Phi}{\partial X} + \frac{\partial\Phi}{\partial T}\right), \tag{A2}$$

where the relationship between (X,T) and (x,t) is given in (5).

The governing Eq. (3) can be re-written as

$$\Phi_{XX} + \frac{\xi^2}{\alpha^2}\Phi_{ZZ} = 0. \tag{A3}$$

With the boundary condition (4b and 4c), and applying the perturbation approximation (6), we have

$$\Phi_0 = H_0, \tag{A4a}$$

$$\Phi_1 = H_1, \tag{A4b}$$

$$\Phi_2 = H_2 + \frac{(H^2 - Z^2)}{2\alpha^2}H_{0XX}, \tag{A4c}$$

$$\Phi_3 = H_3 + \frac{(H^2 - Z^2)}{2\alpha^2}H_{1XX}. \tag{A4d}$$

Substituting (A4) into the non-dimensional form of (4d), we have

$$O(1) : 2H_{0T} = (H_0 H_{0X})_X, \tag{A5a}$$

$$O(\xi) : 2[H_{1T} + \sin(T)\cot(\beta)H_{0X}] = (H_0 H_1)_{XX}, \tag{A5b}$$

with non-dimensional boundary conditions generated from (4a) and (4e):

$$H_0(0,T) = 1 + \alpha\cos(T),\ H_1(0,T) = 0, \tag{A6a}$$

$$H_{0X}(\infty,T) = H_{1X}(\infty,T) = 0. \tag{A6b}$$

in which (A5) and (A6) are identical to (7) and (8a and 8b).

References

Barry, D.A., Barry, S.J., Parlange, J.-Y., 1996. Capillarity correction to periodic solutions of the shallow flow approximation, in: Pattiaratchi, C.B. (Ed.), Mixing Processes in Estuaries and Coastal Seas Coastal and Estuarine Studies, vol. 50. American Geophysical Union, Washington, DC, pp. 496–510.

Bear, J., 1972. Dynamics of Fluids in Porous Media. American Elsevier, New York.

Cartwright, N., Neilsen, P., Dunn, S., 2003. Watertable waves in an unconfined aquifer: experiments and modeling. Water Resources Research 39, 1330 doi:10.1029/2003 WR002185.

Cartwright, N., Nielsen, P., Li, L., 2004. Experimental observations of water table waves in an unconfined aquifer with a sloping boundary. Advances in Water Resources 27, 991–1004.

Dagan, D., 1967. Second-order theory of shallow free-surface flow in porous media. Quarterly Journal of Mechanics and Applied Mathematics 20, 517–526.

Friedrichs, K.O., 1948. On the derivation of the shallow water theory (Appendix to a paper by J.J. Stoker). Communications on Pure and Applied Mathematics 1, 81–85.

Kevorkian, J., Cole, J.D., 1981. Perturbation Methods in Applied Mathematics. Springer-Verlag, Berlin.

Li, L., Barry, D.A., Stagnitti, F., Parlange, J.-Y., Jeng, D.-S., 2000. Beach water table fluctuations due to spring-neap tides: moving boundary effects. Advances in Water Resources 23, 817–824.

Nielsen, P., 1990. Tidal dynamics of the water table in beaches. Water Resources Research 26 (9), 2127–2134.

Nielsen, P., Aseervatham, R., Fenton, J.D., Perrochet, P., 1997. Groundwater waves in aquifers of intermediate depths. Advances in Water Resources 20 (1), 37–43.

Parlange, J.-Y., Stagnitti, F., Starr, J.L., Braddock, R.D., 1984. Free-surface flow in porous media and periodic solution of the shallow flow approximation. Journal of Hydrology 70, 251–263.

Teo, H.T., Jeng, D.-S., Seymour, B.R., Barry, D.A., Li, L., 2003. A new analytical solution for water table fluctuations in coastal aquifers with sloping beaches. Advances in Water Resources 26, 1239–1247.