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New approximation for free surface flow of groundwater: capillarity correction

D.-S. Jeng [a,*], B.R. Seymour [b], D.A. Barry [c], L. Li [d], J.-Y. Parlange [e]

[a] Department of Civil Engineering, The University of Sydney, NSW 2006, Australia
[b] Department of Mathematics, The University of British Columbia, Vancouver, Canada V6T 1Z2
[c] Contaminated Land Assessment and Remediation Research Centre, Institute for Infrastructure and Environment, School of Engineering and Electronics, The University of Edinburgh, Edinburgh EH9 3JL, United Kingdom
[d] School of Engineering, The University of Queensland 4072, Australia
[e] Department of Biological and Environmental Engineering, Cornell University, Ithaca, NY 14853-5701, USA

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Abstract

An existing capillarity correction for free surface groundwater flow as modelled by the Boussinesq equation is re-investigated. Existing solutions, based on the shallow flow expansion, have considered only the zeroth-order approximation. Here, a second-order capillarity correction to tide-induced watertable fluctuations in a coastal aquifer adjacent to a sloping beach is derived. A new definition of the capillarity correction is proposed for small capillary fringes, and a simplified solution is derived. Comparisons of the two models show that the simplified model can be used in most cases. The significant effects of higher-order capillarity corrections on tidal fluctuations in a sloping beach are also demonstrated.

Keywords: Hydraulic conductivity; Sloping beach; Capillary fringe; Tidal forcing; Shallow flow expansion; Simplified approximation

1. Introduction

Groundwater heads in coastal aquifers fluctuate in response to oceanic tides. Such fluctuations are particularly important for erosion control, saltwater intrusion and chemical transformation [4,7]. To simplify the problem, most previous studies of groundwater hydraulics in coastal aquifers have been based on the zeroth-order shallow flow expansion, i.e., the Boussinesq equation with the Dupuit assumption. For example, Dagan [3] and Parlange et al. [9] derived analytical solutions for a vertical beach. They demonstrated the importance of higher-order components in tide-induced watertable fluctuations in a coastal aquifer.

In the field, a sloping sandy beach is more realistic than a vertical beach. Nielsen [8] appears to be the first to derive a higher-order analytical solution for watertable fluctuations in a sloping beach. Later, Nielsen's solution was applied to analyse field measurements [11]. However, Nielsen's solution did not satisfy the boundary condition at the intersection of oceanic and inland water bodies. Later, Li et al. [6] proposed a new apprximation incorporating the concept of a moving boundary, which satisfies the oceanic/land boundary condition. However, both solutions break down for small beach slopes because the mathematical approach included a perturbation parameter involving the beach slope, which could violate the condition that the perturbation parameter is small. Furthermore, all the aforementioned investigations only considered the solution

* Corresponding author. Fax: +61 2 9351 3343.
E-mail addresses: d.jeng@civil.usyd.edu.au (D.-S. Jeng), seymour@math.ubc.ca (B.R. Seymour), d.a.barry@ed.ac.uk (D.A. Barry), l.li@uq.edu.au (L. Li), jp58@cornell.edu (J.-Y. Parlange).

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up to $O(\alpha^2)$ (where α is the amplitude parameter, defined by the ratio of tidal amplitude to the mean aquifer thickness).

Recently Teo et al. [13] re-examined the shallow flow expansion used by previous researchers, and derived a new analytical solution for tide-induced watertable fluctuations in a sloping beach. Two perturbation parameters, the amplitude parameter (α) and the shallow water parameter (ε), which are defined in (3), were used, thereby generalising and extending previous solutions [6,8]. Their solution [13], which extends the expansion up to $O(\varepsilon^2)$, was shown to markedly affect predictions of watertable fluctuations in coastal aquifers.

In the conventional approach to describing tidal fluctuations in coastal aquifers, it is often assumed that the upper free surface is a sharp boundary between saturated and dry aquifer material. This assumption is an oversimplification in many situations as the upper boundary is not abrupt, but a diffuse transition zone of partially unsaturated material (see Fig. 1). Parlange and Brutsaert [10] proposed a capillarity correction to describe the effect of the diffuse transition zone based on the Boussinesq equation. Later, Barry et al. [1] extended Parlange and Brutsaert's work to order $O(\alpha^2)$, and concluded that the capillarty correction is important at high frequencies. These results were then extended to order $O(\alpha^3)$ by Teo et al. [12], but no significant differences were found between the results for $O(\alpha^2)$ and $O(\alpha^3)$. All these investigations considered only the zeroth-order capillarity correction for a vertical beach, not for a sloping beach.

The aim of this paper is to derive a higher-order capillarity correction to free surface flow of groundwater in a sloping beach. First, based on the capillarity correction proposed by Parlange and Brutsaert [10], a second-order capillarity correction $O(\varepsilon^2)$ is derived. Then a new definition of the capillary fringe is proposed for the case of small capillary number, N_{cap} (defined below), and a simplified model is derived. With the two new analytical solutions, the effects of the capillarity correction

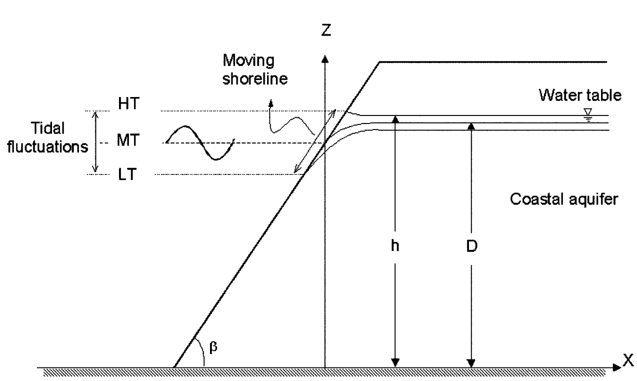

Fig. 1. Schematic diagram of tide-induced watertable fluctuations in a coastal aquifer.

and sloping beaches on watertable fluctuations in coastal aquifers are discussed in detail.

2. Boundary value problem

In this study the flow is assumed to be homogeneous and incompressible in a rigid porous medium. The configuration of the groundwater flow in a coastal aquifer is shown in Fig. 1. In the figure, $h(x,t)$ is the total tide-induced water table height, D is the thickness of the aquifer, and β is the slope of the beach. Seepage face effects are ignored in this study. Since the fluid is incompressible, the free surface flow of groundwater, satisfying the conservation of mass, leads to Laplace's equation for the hydraulic head [2]:

$$\phi_{xx} + \phi_{zz} = 0 \quad 0 \leqslant z \leqslant h(x,t). \tag{1}$$

Eq. (1) is to be solved subject to the following boundary conditions:

$$\phi_z = 0 \quad \text{at } z = 0, \tag{2a}$$

$$\phi = h \quad \text{at } z = h, \tag{2b}$$

$$\phi(x_o(t),t) = D(1+\alpha\cos\omega t), \quad x_o(t) = A\cot\beta\cos\omega t, \tag{2c}$$

$$n_e\phi_t = K\left[\phi_x^2 + \phi_z^2\right] + q - (K+q)\phi_z \quad \text{at } z = h, \tag{2d}$$

$$\phi_x = 0 \quad \text{as } x \to \infty. \tag{2e}$$

Note that the soil properties are defined by the soil porosity (n_e) and hydraulic conductivity (K). In (2d), q is the source term representing the rate at which water crosses the saturated surface.

To simplify the mathematical procedure, we introduce the following non-dimensional variables:

$$\begin{aligned} &X = \frac{x}{L} = \frac{\varepsilon x}{D}, \quad Z = \frac{z}{D}, \quad H = \frac{h}{D}, \\ &\Phi = \frac{\phi}{D}, \quad \alpha = \frac{A}{D}, \\ &T = \omega t, \quad L = \sqrt{\frac{2KD}{n_e\omega}}, \quad \varepsilon = \frac{D}{L} = \sqrt{\frac{n_e\omega D}{2K}}, \\ &q = K\varepsilon^2\Psi. \end{aligned} \tag{3}$$

In (3), two non-dimensional parameters, the shallow water parameter (ε) and the amplitude parameter (α), are introduced. The shallow water parameter (ε) represents the ratio of the water table height (D) to the linear decay length (L). As shown in (3), ε is entirely controlled by the material constants and the boundary condition (2c), and for shallow water flow, $\varepsilon \ll 1$. The amplitude parameter α, representing the ratio of tidal amplitude (A) and mean thickness of aquifer (D), is normally less than 1. Thus, there are three independent parameters defined by the material properties and the boundary conditions, the shallow water parameter, ε, the amplitude parameter, α and the beach slope, β. The approximate solution is constructed under the assumptions of small ε and α, allowing a large range of β (0 < βπ/2).

Since the boundary condition at the intersection of ocean and the beach is on a moving boundary, we define the new independent variables as [6,13]

$$X_1 = X - X_0(T), \quad \text{and} \quad T_1 = T. \tag{4}$$

Then

$$\frac{\partial f}{\partial T} = \frac{\partial f}{\partial T_1} + \frac{\partial f}{\partial X_1}\frac{\partial X_1}{\partial T} = \frac{\partial f}{\partial T_1} + \alpha\varepsilon\cot\beta\sin T_1\frac{\partial f}{\partial X_1} \tag{5}$$

where f is a dependent variable such as Φ, H or Ψ.

To apply the perturbation technique to the non-linear kinematic boundary condition (2d), the watertable height (H), potential head (Φ) and capillary fringe (Ψ) are expressed in powers of the shallow water parameter (ε):

$$H = \sum_{n=0}^{\infty}\varepsilon^n H_n, \quad \Phi = \sum_{n=0}^{\infty}\varepsilon^n \Phi_n \quad \text{and} \quad \Psi = \sum_{n=0}^{\infty}\varepsilon^n \Psi_n, \tag{6}$$

resulting in the following equations to second-order:

$$\mathrm{O}(1): \ 2H_{0T_1} = (H_0 H_{0X_1})_{X_1} + \Psi_0, \tag{7a}$$

$$\mathrm{O}(\varepsilon): \ 2[H_{1T_1} + \alpha\sin(T_1)\cot(\beta)H_{0X_1}] = (H_0H_1)_{X_1X_1} + \Psi_1, \tag{7b}$$

$$\mathrm{O}(\varepsilon^2): \ 2[H_{2T_1} + \alpha\sin(T_1)\cot(\beta)H_{1X_1}] = \frac{1}{2}(H_1^2)_{X_1X_1} + (H_0H_2)_{X_1X_1} + \frac{1}{3}(H_0^3 H_{0X_1X_1})_{X_1X_1} + \Psi_2 + \Psi_0 H_0 H_{0X_1X_1} \tag{7c}$$

with boundary conditions

$$H_0(0,T_1) = 1 + \alpha\cos(T_1), \quad H_1(0,T_1) = H_2(0,T_1) = \cdots = 0, \tag{8a}$$

$$H_{0X_1}(\infty,T_1) = H_{1X_1}(\infty,T_1) = H_{2X_1}(\infty,T_1) = \cdots = 0, \tag{8b}$$

where $\Psi = \Psi_0 + \varepsilon\Psi_1 + \ldots$ are the capillary source contributions that are derived in the next section.

3. Approximation I: complete solution

3.1. Definition of capillarity correction

Following Parlange and Brutsaert [10], the capillary source contribution is defined as

$$q \simeq -\frac{\partial}{\partial t}\left[\frac{\int_{\theta_r}^{\theta_0}(\theta-\theta_r)D\,d\theta}{K + \int_h^{\infty}\frac{\partial\theta}{\partial t}\,dz}\right], \tag{9}$$

where θ_0 is the volumetric water content at saturation and θ_r its residual value. From conservation of mass, we have

$$\int_h^{\infty}\frac{\partial\theta}{\partial t}\,dz = n_e\frac{\partial h}{\partial t} + q. \tag{10}$$

We define the capillary fringe constant [10]

$$B = \int_{\theta_r}^{\theta_o}\frac{(\theta-\theta_r)D}{K}\,d\theta, \tag{11}$$

representing an average suction required to extract water held by capillarity. The importance of capillarity increases with B.

Following Parlange and Brutsaert [10], (9)–(11) then give

$$q = -\frac{\partial}{\partial t}\left[\frac{B}{1 + (hh_x)_x}\right]. \tag{12}$$

Now we introduce non-dimensional and perturbation parameters given in (3) and (5) into (12),

$$\begin{aligned}\Psi^{\mathrm{I}} &= \frac{q}{K\varepsilon^2} = -\frac{B}{K\varepsilon^2}\frac{\partial}{\partial t}\left[\frac{1}{1+(hh_x)_x}\right],\\ &= -\frac{N_{\mathrm{cap}}}{\varepsilon^2}\left\{\frac{\partial}{\partial T_1}\left[\frac{1}{1+\varepsilon^2(HH_{X_1})_{X_1}}\right]\right.\\ &\quad\left. + \alpha\varepsilon\cot\beta\sin(T_1)\frac{\partial}{\partial X_1}\left[\frac{1}{1+\varepsilon^2(HH_{X_1})_{X_1}}\right]\right\},\\ &= \Psi_0^{\mathrm{I}} + \varepsilon\Psi_1^{\mathrm{I}} + \varepsilon^2\Psi_2^{\mathrm{I}} + \mathrm{O}(\varepsilon^3),\end{aligned} \tag{13}$$

where $N_{\mathrm{cap}} = \omega B/K$ is defined as the *capillary number*, which the inverse of the form defined by Li et al. [5]. The superscript "I" denotes the definition proposed by Parlange and Brutsaert [10].

In (13), Ψ_0^{I}, Ψ_1^{I} and Ψ_2^{I} are given by

$$\Psi_0^{\mathrm{I}} = N_{\mathrm{cap}}(H_0H_{0X_1})_{X_1T_1}, \tag{14a}$$

$$\Psi_1^{\mathrm{I}} = N_{\mathrm{cap}}\left[(H_0H_1)_{X_1X_1T_1} + \alpha\cot\beta\sin(T_1)(H_0H_{0X_1})_{X_1X_1}\right], \tag{14b}$$

$$\Psi_2^{\mathrm{I}} = N_{\mathrm{cap}}\left\{(H_0H_2)_{X_1X_1T_1} + (H_1H_{1X_1})_{X_1T_1} - \frac{1}{2}\left[\left((H_0H_{0X_1})_{X_1}\right)^2\right]_{T_1} + \alpha\cot\beta\sin(T_1)(H_0H_1)_{X_1X_1}\right\}. \tag{14c}$$

Note that only the zeroth-order component (Ψ_0^{I}) was presented in Parlange and Brutsaert [10].

3.2. The second-order approximation

The tide-induced water table fluctuation (H) can be expanded in terms of the shallow water and amplitude parameters (ε and α) as

$$H_{mn} = 1 + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \varepsilon^m \alpha^n H_{mn}. \tag{15}$$

Following the procedure in Barry et al. [1] and Teo et al. [13], H_{mn} can be expressed as

$$H_{01}(X_1, T) = \mathrm{e}^{-P_1X_1}\cos\theta_1, \tag{16}$$

$$\begin{aligned} H_{02} = {} & \frac{1}{4}\left[1 - \mathrm{e}^{-2P_1X_1}\right] \\ & + \frac{1+6N_{\mathrm{cap}}^2}{2(1+9N_{\mathrm{cap}}^2)}\left(\mathrm{e}^{-\sqrt{2}P_2X_1}\cos\theta_2 - \mathrm{e}^{-2P_1X_1}\cos 2\theta_1\right) \\ & + \frac{N_{\mathrm{cap}}}{2(1+9N_{\mathrm{cap}}^2)}\left(\mathrm{e}^{-\sqrt{2}P_2X_1}\sin\theta_2 - \mathrm{e}^{-2P_1X_1}\sin 2\theta_1\right). \end{aligned} \tag{17}$$

$$H_{11} = 0 \tag{18}$$

$$\begin{aligned} H_{12} = {} & \frac{\cot\beta}{(P_1^2+Q_1^2)^2}\{(P_1^2+Q_1^2)Q_1 - A_4 \\ & + \mathrm{e}^{-P_1X_1}[(-(P_1^2+Q_1^2)P_1 + A_3)\sin(Q_1X_1) \\ & + ((P_1^2+Q_1^2)Q_1 + A_4)\cos(Q_1X_1)]\} \\ & + \cot(\beta)\{A_1[\mathrm{e}^{-P_1X_1}\cos\theta_3 - \mathrm{e}^{-\sqrt{2}P_2X_1}\cos\theta_2] \\ & - A_2[\mathrm{e}^{-P_1X_1}\sin\theta_3 - \mathrm{e}^{-\sqrt{2}P_2X_1}\sin\theta_2]\} \end{aligned} \tag{19}$$

$$H_{21} = \frac{X_1}{6}\mathrm{e}^{-P_1X_1}(B_1\cos\theta_1 - B_2\sin\theta_1) \tag{20}$$

$$\begin{aligned} H_{22} = {} & \frac{1}{4P_1^3}(B_3 + B_4P_1)(1 - \mathrm{e}^{-2P_1X_1}) \\ & - \frac{1}{4P_1^2}B_3X_1\mathrm{e}^{-2P_1X_1} + \mathrm{e}^{-\sqrt{2}P_2X_1}[(B_9X_1 - B_7)\cos\theta_2 \\ & - (B_{10}X_1 - B_8)\sin\theta_2] + \mathrm{e}^{-2P_1X_1}[(B_5X_1 + B_7)\cos 2\theta_1 \\ & - (B_6X_1 + B_8)\sin 2\theta_1] \end{aligned} \tag{21}$$

where

$$\theta_1 = T_1 - Q_1X_1, \quad \theta_2 = 2T_1 - \sqrt{2}Q_2X_1, \quad \theta_3 = 2T_1 - Q_1X_1$$

and

$$\begin{Bmatrix} P_m \\ Q_m \end{Bmatrix} = \left[\frac{1}{\sqrt{1+m^2N_{\mathrm{cap}}^2}} \pm \frac{mN_{\mathrm{cap}}}{1+m^2N_{\mathrm{cap}}^2}\right]^{1/2}, \quad m = 1, 2. \tag{22}$$

The various coefficients A_i and B_i are given in the Appendix A.

3.3. Special cases

Three special cases can be easily deduced from the above analytical solutions.

3.3.1. Special case I: vertical beach without capillarity correction ($N_{cap} = 0$, $\beta = \pi/2$).

For the simplest case of a vertical beach, $\beta = \pi/2$ and $N_{\mathrm{cap}} = 0$, when $X_1 = X$, $T_1 = T$ and $\cot\beta = 0$, the solution becomes

$$\begin{aligned} H(X,T) = {} & \alpha\mathrm{e}^{-X}\cos\theta_1 + \alpha^2\left\{\frac{1}{4}(1 - \mathrm{e}^{-2X}) \right. \\ & \left. + \frac{1}{2}\left[\mathrm{e}^{-\sqrt{2}X}\cos(\theta_2) - \mathrm{e}^{-2X}\cos(2\theta_1)\right]\right\} \\ & - \frac{\sqrt{2}}{3}\varepsilon^2\alpha X\mathrm{e}^{-X}\cos\left(\theta_1 - \frac{\pi}{4}\right) \\ & + \frac{1}{3}\varepsilon^2\alpha^2\left\{-1 + \left(1 + \frac{X}{2}\right)\mathrm{e}^{-2X} \right. \\ & - 2X\mathrm{e}^{-\sqrt{2}X}\cos\left(\theta_2 - \frac{\pi}{4}\right) + \mathrm{e}^{-\sqrt{2}X}\sin\theta_2 \\ & \left. + \sqrt{2}X\mathrm{e}^{-2X}\cos\left(2\theta_1 - \frac{\pi}{4}\right) - \mathrm{e}^{-2X}\sin 2\theta_1\right\}. \end{aligned} \tag{23}$$

Note that (23) is identical to the O(α^2) solution for a vertical beach given in Parlange et al. [9].

3.3.2. Special case II: vertical beach with capillarity correction ($N_{cap} \neq 0$, $\beta = \pi/2$)

The second special case is for a vertical beach with capillarity correction, i.e., $N_{\mathrm{cap}} \neq 0$, $\beta = \pi/2$, when $X_1 = X$, $T_1 = T$ and $\cot\beta = 0$, the solution is

$$\begin{aligned} H(X,T) = {} & \alpha\mathrm{e}^{-P_1X}\cos\theta_1 + \alpha^2\left[\frac{1}{4}(1 - \mathrm{e}^{-2X}) \right. \\ & + \frac{(1+6N_{\mathrm{cap}}^2)}{2(1+9N_{\mathrm{cap}}^2)}(\mathrm{e}^{-\sqrt{2}P_2X}\cos\theta_2 - \mathrm{e}^{-2P_1X}\cos 2\theta_1) \\ & \left. + \frac{N_{\mathrm{cap}}}{2(1+9N_{\mathrm{cap}}^2)}(\mathrm{e}^{-\sqrt{2}P_2X}\sin\theta_2 - \mathrm{e}^{-2P_1X}\sin 2\theta_1)\right] \\ & + \varepsilon^2\alpha\frac{X}{6}\mathrm{e}^{-P_1X}\{B_1\cos\theta_1 - B_2\sin\theta_1\} \\ & + \varepsilon^2\alpha^2\left\{\frac{1}{4P_1^3}(B_3 + B_4P_1)(1 - \mathrm{e}^{-2P_1X}) \right. \\ & - \frac{1}{4P_1^2}B_3X\mathrm{e}^{-2P_1X} + \mathrm{e}^{-\sqrt{2}P_2X}[(B_9X - B_7)\cos\theta_2 \\ & - (B_{10}X - B_8)\sin\theta_2] + \mathrm{e}^{-2P_1X}[(B_5X + B_7)\cos 2\theta_1 \\ & \left. - (B_6X + B_8)\sin 2\theta_1]\right\}, \end{aligned} \tag{24}$$

Note that (24) is identical to the solution of O(α^2) for a vertical beach with capillary correction given in Barry et al. [1].

3.3.3. Special case III: sloping beach without the capillarity correction ($N_{cap} = 0$, $\beta \leqslant \pi/2$)

The third special case is for a sloping beach with no capillarity correction, i.e., $N_{\text{cap}} = 0$, $\beta \leqslant \pi/2$. From (16)–(21), the solution is

$$\begin{aligned} H(X_1, T_1) &= \alpha e^{-X_1}\cos(\theta_1) + \alpha^2\left[\frac{1}{4}(1 - e^{-2X_1}) + \frac{1}{2}e^{-\sqrt{2}X_1}\cos(\theta_2) - \frac{1}{2}e^{-2X_1}\cos(2\theta_1)\right] \\ &\quad + \frac{1}{\sqrt{2}}\cot(\beta)\varepsilon\alpha^2\left[\frac{1}{\sqrt{2}} - e^{-X_1}\cos\left(X_1 - \frac{1}{4}\pi\right) + e^{-\sqrt{2}X_1}\cos\left(\theta_2 + \frac{1}{4}\pi\right) - e^{-X_1}\cos\left(\theta_3 + \frac{1}{4}\pi\right)\right] \\ &\quad - \frac{\sqrt{2}}{3}\varepsilon^2\alpha X_1 e^{-X_1}\cos\left(\theta_1 - \frac{\pi}{4}\right) + \frac{1}{3}\varepsilon^2\alpha^2 \\ &\quad \times\left\{-1 + \left(1 + \frac{X_1}{2}\right)e^{-2X_1} - 2X_1 e^{-\sqrt{2}X_1}\cos\left(\theta_2 - \frac{\pi}{4}\right) + e^{-\sqrt{2}X_1}\sin(\theta_2) + \sqrt{2}X_1 e^{-2X_1}\cos\left(2\theta_1 - \frac{\pi}{4}\right) - e^{-2X_1}\sin(2\theta_1)\right\}, \end{aligned} \tag{25}$$

which is identical to that derived by Teo et al. [13].

3.4. Effects of higher-order components

As mentioned previously, existing analytical solutions which include the capillarity correction in approximating the tide-induced watertable fluctuations in coastal aquifers have been up to $O(\varepsilon^0\alpha^3)$ for a vertical beach [1,12]. Using the new higher-order solutions given here, it is possible to examine directly the effects of higher-order components. A straightforward way to compare orders is to calculate the watertable level by using the above approximation. Watertable levels (H) versus ($T/2\pi$) for various orders are plotted in Fig. 2. Significant differences between the zeroth-order $O(\alpha)$ and second-order solution ($\varepsilon^2\alpha^2$) are observed.

In Fig. 2, two different beach slopes are considered, $\beta = \pi/6$ and $\pi/3$. It is observed that a large beach slope will enhance the influence of the higher-order components on the watertable height (H). For example, the maximum difference between the linear solution, $O(\alpha)$, and the second-order solution, $O(\varepsilon^2\alpha^2)$, is 2.5% of D (the mean aquifer thickness) with $\beta = \pi/6$, while it is 5% of D with $\beta = \pi/3$. This results implies that non-linear effects are more important for steeper beaches.

3.5. Effects of the capillarity correction

The importance of the capillarity correction is quantified by the capillary number, N_{cap} ($=\omega B/K$). The capillary number increases with the frequency and capillary fringe length, and as the hydraulic conductivity decreases. In most coastal aquifers, the capillary fringe constant (B) varies from 0 to 0.4 m and the hydraulic conductivity (K) varies from 10 m/d to 1000 m/d [2]. We consider tides, the frequency of which (ω) is 4π d^{-1}. Thus, the capillary number, N_{cap}, varies between 0 and 0.5. Fig. 3 demonstrates the importance of the capillary number on the watertable fluctuations. This indicates that the influence of the capillary correction is about 2% of watertable level H.

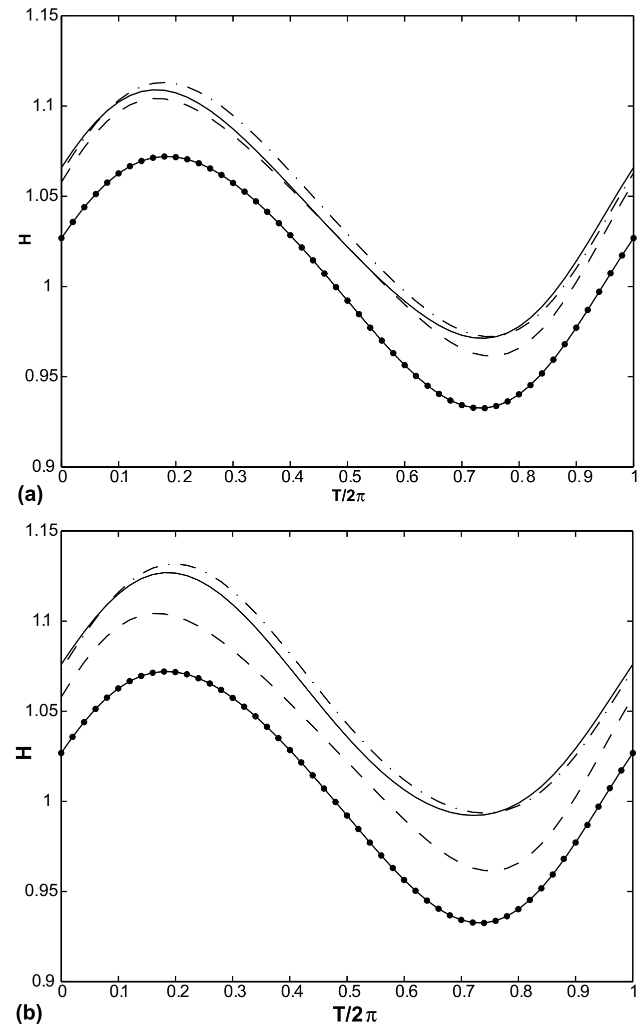

Fig. 2. Distribution of watertable fluctuations (H) versus time ($T/2\pi$) for various order solutions and beach slopes: (a) $\beta = \pi/6$ and (b) $\beta = \pi/3$ ($\varepsilon = \alpha = 0.35$, and $X = 1.5$). (–•–) the $O(\alpha)$ solution, (–) the $O(\alpha^2)$ solution; (-·) the $O(\varepsilon\alpha^2)$ solution and (–) the $O(\varepsilon^2\alpha^2)$ solution.

4. Approximation II: simplified solution

4.1. New definition of capillarity correction

Here, we consider the situation when the capillary source term is much smaller than other terms, i.e., $q \ll n_e\frac{\partial h}{\partial t}$. Then, (10) can be written as

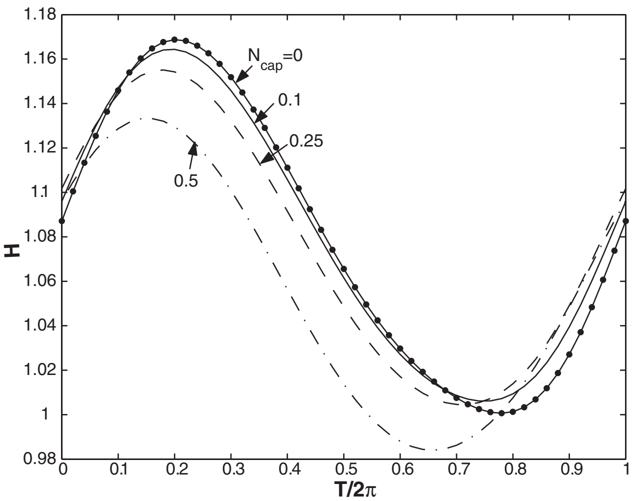

Fig. 3. Distribution of watertable fluctuations (H) versus time ($T/2\pi$) for various values of N_{cap} ($\epsilon = \alpha = 0.4$, $X = 1.5$, $\beta = \pi/6$).

$$\int_h^\infty \frac{\partial\theta}{\partial t}\,dz = n_e\frac{\partial h}{\partial t} \tag{26}$$

and (9) becomes

$$q \simeq -\frac{\partial}{\partial t}\left[\frac{B}{1+\frac{n_e}{K}\frac{\partial h}{\partial t}}\right]. \tag{27}$$

Thus,

$$\begin{aligned}\Psi^{\mathrm{II}} &= \frac{q}{K\varepsilon^2} = -\frac{B}{K\varepsilon^2}\frac{\partial}{\partial t}\left(\frac{1}{1+\frac{n_e}{K}\frac{\partial h}{\partial t}}\right)\\ &= N_{cap}\{2H_{0T_1T_1} + \varepsilon[2H_{1T_1T_1} + \alpha\cot\beta(H_{0X_1T_1}\sin T_1\\ &\quad + H_{0X_1}\cos T_1) + 2\alpha\cot\beta(H_{0X_1T_1}\sin T_1)]\\ &\quad + \varepsilon^2[2H_{2T_1T_1} + \alpha\cot\beta(H_{1X_1T_1}\sin T_1 + H_{1X_1}\cos T_1)\\ &\quad - 2(H^2_{0T_1})_{T_1} + 2\alpha\cot\beta\sin T_1(H_{1X_1T_1}\\ &\quad + \alpha\sin T_1\cot\beta H_{0X_1X_1})]\} + O(\varepsilon^3)\\ &= \Psi_0^{\mathrm{II}} + \varepsilon\Psi_1^{\mathrm{II}} + \varepsilon^2\Psi_2^{\mathrm{II}} + O(\varepsilon^3),\end{aligned} \tag{28}$$

where

$$\Psi_0^{\mathrm{II}} = 2N_{cap}H_{0T_1T_1} \tag{29a}$$

$$\begin{aligned}\Psi_1^{\mathrm{II}} &= N_{cap}[2H_{1T_1T_1} + \alpha\cot\beta(H_{0X_1T_1}\sin T_1\\ &\quad + H_{0X_1}\cos T_1) + 2\alpha\cot\beta H_{0X_1T_1}\sin T_1]\end{aligned} \tag{29b}$$

$$\begin{aligned}\Psi_2^{\mathrm{II}} &= N_{cap}[2H_{2T_1T_1} + \alpha\cot\beta(H_{1X_1T_1}\sin T_1\\ &\quad + H_{1X_1}\cos T_1) - 2(H^2_{0T_1})_{T_1} + 2\alpha\cot\beta\sin T_1(H_{1X_1T_1}\\ &\quad + \alpha\sin T_1\cot\beta H_{0X_1X_1})]\end{aligned} \tag{29c}$$

in which the superscript "II" denotes the new definition.

4.2. Simplified solution

The terms in the simplified solution can be written as

$$H_{01} = e^{-R_1X_1}\cos\delta_1, \tag{30}$$

$$\begin{aligned}H_{02} &= \frac{1}{4}(1 - e^{-2R_1X_1})\\ &\quad + \frac{1}{2}\left(e^{-\sqrt{2}R_2X_1}\cos\delta_2 - e^{-2R_1X_1}\cos 2\delta_1\right)\\ &\quad + \frac{N_{cap}}{2}\left(e^{-\sqrt{2}R_2X_1}\sin\delta_2 - e^{-2R_1X_1}\sin 2\delta_1\right),\end{aligned} \tag{31}$$

$$H_{11} = 0, \tag{32}$$

$$\begin{aligned}H_{12} &= \frac{\cot\beta}{(R_1^2+S_1^2)}\left\{S_1 - \frac{N_{cap}}{2}R_1(1+3S_1) - e^{-R_1X_1}\right.\\ &\quad \times\left[(R_1\sin(S_1X_1) + S_1\cos(S_1X_1))\right.\\ &\quad \left.\left.+\frac{N_{cap}}{2}(1+3S_1)(S_1\sin(S_1X_1) - R_1\cos(S_1X_1))\right]\right\}\\ &\quad + \cot\beta[C_1(e^{-R_1X_1}\cos\delta_3 - e^{-\sqrt{2}R_2X_1}\cos\delta_2)\\ &\quad + C_2(e^{-\sqrt{2}R_2X_1}\sin\delta_2 - e^{-R_1X_1}\sin\delta_3)],\end{aligned} \tag{33}$$

$$H_{21} = \frac{X_1}{6}e^{-R_1X_1}(D_1\cos\delta_1 + D_2\sin\delta_1), \tag{34}$$

and

$$\begin{aligned}H_{22} &= \frac{1}{4R_1^3}(D_3 + D_4R_1)(1 - e^{-2R_1X_1}) - \frac{1}{4R_1^2}D_3X_1e^{-2R_1X_1}\\ &\quad + e^{-\sqrt{2}R_2X_1}[(D_9X_1 - D_7)\cos\delta_2\\ &\quad - (D_{10}X_1 - D_8)\sin\delta_2]\\ &\quad + e^{-2R_1X_1}[(D_5X_1 + D_7)\cos 2\delta_1\\ &\quad - (D_6X_1 + D_8)\sin 2\delta_1],\end{aligned} \tag{35}$$

where

$$\delta_1 = T_1 - S_1X_1,\quad \delta_2 = 2T_1 - \sqrt{2}S_2X_1,\quad \delta_3 = 2T_1 - S_1X_1$$

and

$$\left\{\begin{matrix}R_m\\ S_m\end{matrix}\right\} = \left[\sqrt{1+m^2N_{cap}^2} \pm mN_{cap}\right]^{1/2}. \tag{36}$$

The simplified solution, (32)–(35), can also be reduced to the three special cases through the same procedure as the previous section.

4.3. Comparison of solutions

The new definition of the capillarity correction (i.e., (27)) provides a simplified solution compared with that in Parlange and Brutsaert [10]. We now investigate the difference of watertable fluctuations calculated from the two solutions. In the example, we vary the two perturbation parameters, the shallow water parameter (ε) and the amplitude parameter (α), between 0 and 0.4, this range being reasonable since they are assumed to be significantly less than unity. In the following sections, the

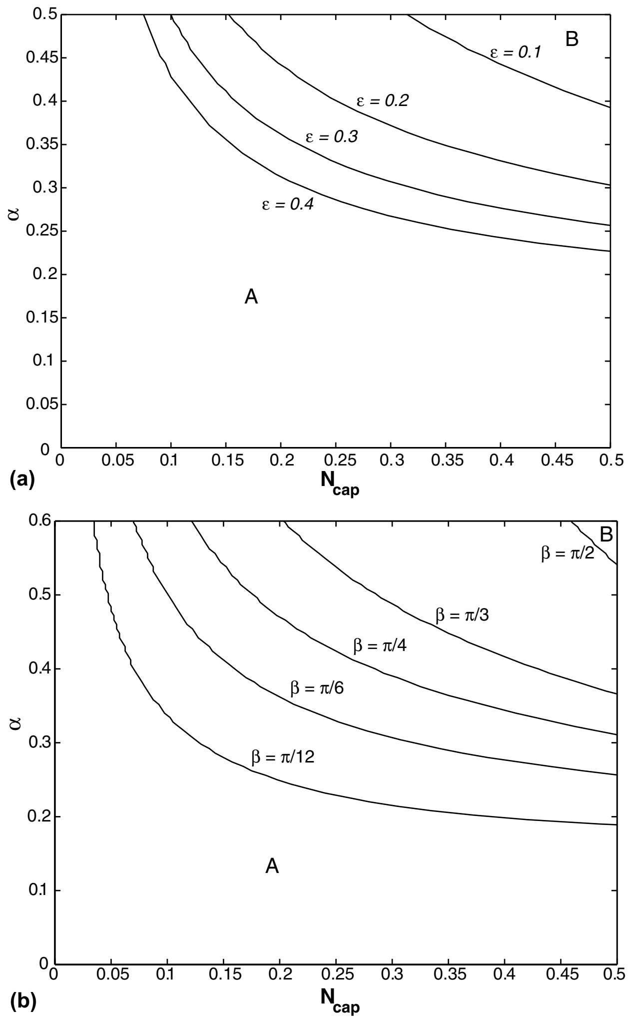

Fig. 4. Applicable zone of the simplified model based on 1.5% of relative differences $(H^{\mathrm{I}} - H^{\mathrm{II}})/H^{\mathrm{I}}$. (a) various ε with $\beta = \pi/6$ and (b) various beach slope β with $\varepsilon = 0.3$. ($T = 0$ and $X = 1.5$).

watertable level calculated from the first approximation, extended from the definition proposed in Parlange and Brutsaert [10], is denoted as H^{I}, whilst the results from the second approximation, the simplified model, is denoted as H^{II}.

To investigate the difference between two solutions H^{I} and H^{II}, given by (12) and (27), respectively. We consider the relative difference $|H^{\mathrm{I}} - H^{\mathrm{II}}|/H^{\mathrm{I}} = 1.5\%$ as an acceptable accuracy. Fig. 4(a) illustrates the applicable zone of the simplified model for various ε with a 1.5% relative difference. In the figure, three parameters, ε, α and N_{cap} vary within reasonable physical ranges as discussed above. For cases belonging to Zone A (below the curves), the simplified model can be used, while the original model should be used for parameters in the range above the curves, in Zone B. The figure clearly indicates that the simplified model is applicable for most cases.

Fig. 4(b) further investigates the applicable zone of the simplified model for various beach slopes. The results demonstrate that the simplified model is more applicable for larger beach slopes. For a vertical beach, the simplified model can replace the original model for the range up to $(\varepsilon, \alpha) = (0.5, 0.5)$.

5. Conclusions

In this paper, the definition of capillarity correction proposed by Parlange and Brutsaert [10] was extended to second-order and to sloping beaches. The significant effects of higher-order components on the watertable fluctuations are also demonstrated. The results indicate the influence of the capillary correction may reach 2% of the watertable level.

A new definition of capillary correction was then proposed for the case of small capillary fringe, and a simplified solution was derived. A comparison of the two solutions indicates that the simplified model can be used in most cases.

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Appendix A. List of coefficients

The coefficients A_i and B_i in (18)–(21) are

$$A_1 = \frac{1}{\Delta}\left[\left(Q_1 - \frac{N_{\mathrm{cap}}}{2}Q_1(Q_1^2 - 3P_1^2)\right)(2P_1Q_1N_{\mathrm{cap}} - (P_1^2 - Q_1^2)) + \left(-P_1 + \frac{N_{\mathrm{cap}}}{2}P_1(P_1^2 - 3Q_1^2)\right) \times (4 - 2P_1Q_1 - N_{\mathrm{cap}}(P_1^2 - Q_1^2))\right] \quad \text{(A.1a)}$$

$$A_2 = \frac{1}{\Delta}\left[\left(-P_1 + \frac{N_{\mathrm{cap}}}{2}P_1(P_1^2 - 3Q_1^2)\right)(2P_1Q_1N_{\mathrm{cap}} - (P_1^2 - Q_1^2)) - \left(Q_1 - \frac{N_{\mathrm{cap}}}{2}Q_1(Q_1^2 - 3P_1^2)\right) \times \left(4 - 2P_1Q_1 - N_{\mathrm{cap}}(P_1^2 - Q_1^2)\right)\right] \quad \text{(A.1b)}$$

$$\Delta = [(P_1^2 - Q_1^2) - 2N_{\mathrm{cap}}P_1Q_1]^2 + [4 - 2P_1Q_1 - 2N_{\mathrm{cap}}(P_1^2 - Q_1^2)]^2 \quad \text{(A.1c)}$$

$$A_3 = \frac{N_{\mathrm{cap}}}{2}P_1(P_1^4 - 10P_1^2Q_1^2 + 5Q_1^4), \quad \text{(A.2a)}$$

$$A_4 = \frac{N_{\text{cap}}}{2} Q_1(Q_1^4 - 10P_1^2 Q_1^2 + 5P_1^4) \tag{A.2b}$$

$$B_1 = (P_1^3 - N_{\text{cap}} Q_1^3) + 3P_1 Q_1 (N_{\text{cap}} P_1 - Q_1), \tag{A.3a}$$

$$B_2 = -(N_{\text{cap}} P_1^3 + Q_1^3) + 3P_1 Q_1 (P_1 + N_{\text{cap}} Q_1), \tag{A.3b}$$

$$B_3 = \frac{1}{3} P_1^2 B_1, \tag{A.3c}$$

$$B_4 = \frac{2}{3} P_1^4 - \frac{1}{3} P_1 B_1 - 2P_1^2 Q_1^2 \tag{A.3d}$$

$$B_5 + iB_6 = \frac{(1 + N_{\text{cap}}^2)(1 + 2iN_{\text{cap}})(P_1 + iQ_1)^2 (B_1 + iB_2)}{12\left[-i(1 + 3N_{\text{cap}}^2) + 2N_{\text{cap}}\right]} \tag{A.4a}$$

$$B_7 + iB_8 = \frac{(1 + N_{\text{cap}}^2)(B_{11} - 4(B_5 + iB_6)(P_1 + iQ_1)(1 + 2iN_{\text{cap}}))}{4\left[-i(1 + 3N_{\text{cap}}^2) + 2N_{\text{cap}}\right]} \tag{A.4b}$$

$$B_9 + iB_{10} = \frac{\left[\left(1 + 6N_{\text{cap}}^2\right) - iN_{\text{cap}}\right]}{3\sqrt{2}(1 + 2iN_{\text{cap}})(1 + 9N_{\text{cap}}^2)} (P_2 + iQ_2)^3 \tag{A.4c}$$

$$B_{11} = -\frac{1}{3}(1 + 2iN_{\text{cap}})(P_1 + iQ_1)(B_1 + iB_2) + 2(P_1 + iQ_1)^4 - \frac{8\left[\left(1 + 6N_{\text{cap}}^2\right) - iN_{\text{cap}}\right]}{3(1 + 9N_{\text{cap}}^2)} (P_1 + iQ_1)^4 \tag{A.4d}$$

The coefficients C_i and D_i in (32)–(35) are

$$C_1 = \frac{1}{2(1 + 9N_{\text{cap}}^2)} \left\{ 3N_{\text{cap}} \left[S_1 - \frac{N_{\text{cap}}}{2} R_1 (1 - 3S_1) \right] - \left[R_1 - \frac{N_{\text{cap}}}{2} S_1 (1 - 3S_1) \right] \right\} \tag{A.5a}$$

$$C_2 = \frac{1}{2(1 + 9N_{\text{cap}}^2)} \left\{ -\left[S_1 - \frac{N_{\text{cap}}}{2} R_1 (1 - 3S_1) \right] + 3N_{\text{cap}} \left[R_1 - \frac{N_{\text{cap}}}{2} S_1 (1 - 3S_1) \right] \right\} \tag{A.5b}$$

$$D_1 = R_1 (R_1^2 - 3S_1^2), \tag{A.6a}$$

$$D_2 = S_1 (S_1^2 - 3R_1^2), \tag{A.6b}$$

$$D_3 = \frac{1}{3} R_1^2 D_1, \tag{A.6c}$$

$$D_4 = \frac{2}{3} R_1^4 - \frac{1}{3} R_1 D_1 - 2R_1^2 S_1^2 - N_{\text{cap}} (R_1^2 - S_1^2) \tag{A.6d}$$

$$D_5 + iD_6 = \frac{i}{12} (R_1 + iS_1)^2 (D_1 - iD_2) \tag{A.7a}$$

$$D_7 + iD_8 = \frac{3iC_4^* + (R_1 + iS_1)^3 (D_1 - iD_2)}{12} \tag{A.7b}$$

$$D_9 + iD_{10} = \frac{(1 - iN_{\text{cap}})}{3\sqrt{2}} (R_2 + iS_2)^3 \tag{A.7c}$$

$$D_{11} = -\frac{1}{3}(R_1 + iS_1)(D_1 - iD_2) + \frac{1}{3}(-2 + 8iN_{\text{cap}})(R_1 + iS_1)^4 - 2N_{\text{cap}}^2 \tag{A.7d}$$

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