Available online at www.sciencedirect.com

SCIENCE DIRECT®

Advances in Water Resources 28 (2005) 1028–1031

Advances in Water Resources

www.elsevier.com/locate/advwatres

Interpolation between Darcy–Weisbach and Darcy for laminar and turbulent flows

W.L. Hogarth [a,*], J.-Y. Parlange [b], C.W. Rose [c], C. Fuentes [d], R. Haverkamp [e], M.T. Walter [b]

[a] Faculty of Science and Information Technology, The University of Newcastle, Callaghan, NSW 2308, Australia
[b] Department of Biological and Environmental Engineering, Riley-Robb Hall, Cornell University, Ithaca, NY 14853, USA
[c] Faculty of Environmental Sciences, Griffith University, Qld. 4111, Australia
[d] Instituto Mexicano de Technologia del Agua, Paseo Cuauhnahuac No. 8532, Col. Progreso, 62550 Jiutepec, Morelos, Mexico
[e] Laboratoire d'Etude des Transferts en Hydrologie et Environnement, UMR (CNRS, INPG, UJF) 5564, LTHE, BP 53, 38041 Grenoble Cedex 9, France

Received 11 March 2004; received in revised form 17 August 2004; accepted 13 September 2004
Available online 19 January 2005

Abstract

An equation describing flow in an open channel with obstacles is derived, following the conservation of momentum approach used by Bélanger and St. Venant. When the obstacles are all submerged the result yields the Darcy–Weisbach equation for turbulent flow in pipes and open channels. When the obstacles are only partially submerged the result leads to the governing equation in a porous medium. If the flow is turbulent the square of the velocity is proportional to the hydraulic gradient and if the flow is laminar, which is the usual case, the velocity is proportional to the hydraulic gradient. This last result is in agreement with Darcy's law in porous media. Thus our equation interpolates between and reduces to, the two fundamental results of Darcy. In general our equation should prove useful in practice for open flow in a channel with both submerged and emerging obstacles.

Keywords: Darcy; Darcy–Weisbach; Porous media; Open channels; Turbulent flow; Laminar flow

1. Introduction

Grassed waterways are a standard means to carry surface runoff while avoiding erosion [1]. A similar situation arises with mulch and other debris in fields or channels. Modern modeling of such flows relies on the use of the appropriate Darcy–Weisbach equation, e.g. see [2–4], which states that the flow velocity, V, is proportional to the square root of the flow depth, D, and the slope, S_o, which we consider to be always small in the following.

This famous formula was obtained by Darcy [5] from his careful measurement of turbulent flows in pipes and then extended to open channels, primarily by Darcy and Bazin [6]. Weisbach [7] obtained the same result, hence the joint name given to the result. Interestingly, Chézy [8] and Du Buat [9] obtained basically the same equation much earlier. Chézy's work was written a few years prior to Du Buat's, but as an internal report only. Based on Du Buat's and others' experiments, Prony [10] suggested a more general formula for pipes and open channels,

$$DS_o = aV + bV^2, \tag{1}$$

with a and b constant, suggesting already that the use of a fractional power would be slightly more accurate in some cases [11]. However if we write

* Corresponding author.
E-mail addresses: bill.hogarth@newcastle.edu.au (W.L. Hogarth), jp58@cornell.edu (J.-Y. Parlange), c.rose@griffith.edu.au (C.W. Rose), cfuentes@tlaloc.imta.mx (C. Fuentes), randel.haverkamp@hmg.inpg.fr (R. Haverkamp).

0309-1708/$ - see front matter
doi:10.1016/j.advwatres.2004.10.012

$$\frac{D^3 S_o g}{\nu^2} = f\left(\frac{DV}{\nu}\right), \tag{2}$$

where g is the acceleration of gravity and ν the kinematic viscosity, which is essentially the expression in [11], with g added to be dimensionally correct, then we can consider a number of possibilities (see also [12] for a more complete discussion). If f is linear we obtain the well known Poiseuille law [13] for laminar flow, whereas if f is quadratic the Darcy–Weisbach result follows. No function, f, will yield DS_o as a function of a fractional power of V as possibly suggested by Eq. (1).

For pipes of relatively small sections taking a function, f, proportional to $(DV/\nu)^\alpha$, was shown by Reynolds [14] to be best with $\alpha \approx 1.75$, i.e. less than $\alpha = 2$ obtained by Darcy–Weisbach, the latter being preferable for the open channels which concerns us here. There is no value of the α that would lead to Manning's well known equation [15], but it is interesting that if we take $\alpha = 1.8$, i.e. close to Reynold's value of 1.753, and replace S_o in Eq. (2) by $S_o^{0.9}$, then Manning's result follows. That 0.9 power being unjustified, theoretically at least, it is preferable to use either Darcy–Weisbach in open channel or Reynolds in small tubes, for turbulent flow, and of course Poiseuille for laminar flow.

Bélanger [16] considered the steady flow in an open channel with slope S_o, which we take as small, and obtained the fundamental result, see [11] for a thorough discussion of Bélanger's contribution, which can be written as,

$$\frac{dD}{dx} = \frac{S_o - S_e}{1 - F^2}, \tag{3}$$

where F is the Froude number,

$$F^2 = \frac{q^2}{gD^3} = \frac{V^2}{gD}, \tag{4}$$

where $q = VD$ is the flux. S_e is now called the friction slope, such that when $S_o = S_e$, $dD/dx = 0$ which, for instance for turbulent flow must reduce to Darcy–Weisbach. Thus with $\alpha = 2$ i.e. $f = (DV/\nu)^2$ in Eq. (2) then we obtain

$$D^3 S_o g = \frac{C_s}{2} D^2 V^2, \tag{5}$$

$C_s/2$ being the constant of proportionality in Eq. (2). When $dD/dx = 0$, $S_e = S_o$ and hence

$$S_e = \frac{C_s V^2}{2gD}. \tag{6}$$

Of course, at the time Bélanger used it Eq. (6) which we call the Darcy–Weisbach equation, was based on Du Buat's results.

In the following, like Bélanger, we shall consider steady flows only, as the laws of resistance to flow, e.g. Darcy–Weisbach, play the same role whether the flow is steady or not. Since we are interested in the laws of resistance to flow we can consider steady state flows only without loss of generality. Of course generalizing Bélanger's results to unsteady flows was a fundamental contribution of St. Venant [17] and Eq. (3) can then be seen as the steady state limit of the St. Venant equations, and as a result is often referred to him, rather than Bélanger.

In the following we shall generalize Eq. (3) when both submerged and/or partially submerged obstacles are present in the channel. As the density of obstacles increases we shall obtain, in the limit, Boussinesq's equation [18] for groundwater flow when the Dupuit–Forchheimer approximation holds [19], [20]. In this case the Boussinesq equation is based on Darcy's other fundamental work on flow in porous media [21].

Much of the above historical background can be found, with additional details, in [11,22] as well as other historical reviews.

2. Interpolation formula and discussion

We essentially follow Bélanger's approach [11,16] taking into account the additional resistance caused by submerged and partially submerged obstacles. With the presence of obstacles the volume they occupy in the flow can be significant [3] and we call θ the porosity, i.e. the volume of water/[volume of water + volume of partially submerged obstacles]. If we call τ the shear stress experienced by the flow, it must be balanced by the weight of water, the momentum and the pressure, or

$$\tau = \rho g S_o D\theta + \rho V^2 \theta \frac{dD}{dx} - \rho g D\theta \frac{dD}{dx}, \tag{7}$$

where ρ is the density of the liquid and we replace the stress by the friction slope, with the identity

$$\tau \equiv \rho g S_e D, \tag{8}$$

S_e must take into account the soil surface and obstacles.

If there was no partially submerged obstacle, Eq. (6) would hold with a modified C_s to take into account submerged obstacles. Indeed this is the standard approach [2–4] used even when emerging obstacles were clearly present. Here we write separately the effect of emerging obstacles, or,

$$S_e = \frac{V^2}{2gD}[C_s\theta + C_d A(1-\theta)], \tag{9}$$

where A represents the ratio of the frontal area of the emerging obstacle, and the cross section of the obstacle

$$A = \frac{4Dd}{\pi d^2}, \tag{10}$$

with d the characteristic diameter of the obstacle. The coefficients θ and $(1 - \theta)$ represent the proportion of surfaces on which each stress is acting. For given obstacles and a turbulent flow C_d and C_s should be constants for given obstacles. Eqs. (7)–(9) complete the formulation of the problem, or

$$\frac{dD}{dx} = \frac{S_o - \frac{S_e}{\theta}}{1 - F^2}, \tag{11}$$

which is, formally, almost identical to Eq. (3). If C_d is neglected in Eq. (9) the result is consistent with the Darcy–Weisbach equation [3–5]. If on the other hand C_s is neglected compared to the effect of C_d we obtain on equation very similar to that used in [23] and [24] for emerging obstacles only (with minor discrepancies for the effect of θ, which were not important there as θ was very close to 1).

We can now show that if the emerging obstacles become more numerous and form a porous medium Eq. (11) leads to Darcy's law for porous media [21]. In that limit θ will be significantly less than one and F^2 will always be negligible in Eq. (11), which reduces to

$$\frac{S_e}{\theta} = S_o - \frac{dD}{dx}, \tag{12}$$

and from Eq. (9), ignoring the C_s term,

$$V^2 = \frac{\pi\theta d}{(1-\theta)2C_d} g\left(S_o - \frac{dD}{dx}\right), \tag{13}$$

where $(S_o - dD/dx)$ is the hydraulic gradient. For fully turbulent flow our formulation gives V^2 proportional to the hydraulic gradient, which is the way Darcy's law should be written [25]. Of course, in a porous medium the Reynolds's number, Vd/v, is normally very small and the flow is laminar and the drag C_d can then be written as

$$C_d = \frac{C_1 v}{Vd} \tag{14}$$

to be consistent with Eq. (2) for a Poiseuille flow, then Eq. (13) becomes,

$$V = \frac{\pi d^2\theta}{2(1-\theta)vC_1} g\left(S_o - \frac{dD}{dx}\right), \tag{15}$$

which is in agreement with Poiseuille's result. This of course is the standard Darcy's equation for laminar flows in porous media [21]. Interestingly this equation also provides an estimate of the saturated conductivity

$$K = \frac{\pi d^2\theta g}{2(1-\theta)vC_1}, \tag{16}$$

where C_1 is a characteristic of the porous medium and should be of order one.

3. Conclusion

Starting with the equation for an open channel turbulent flow, which is based on the Darcy–Weisbach equation, we added the resistance due to obstacles. Submerged obstacles are still consistent with a Darcy–Weisbach equation but with increased friction. This approach is standard and has been used in practice. With emerging obstacles only we improved an earlier expression and the result is consistent with Darcy's law in porous media for laminar flow, which was the case considered by Darcy. But our result also yields the extension of Darcy's law for turbulent flows. Thus by keeping the resistances of both submerged and emerging obstacles we obtained a general equation which covers the limiting results described by Darcy–Weisbach in open channel and by Darcy's law in porous media. Our equation should be useful in practice when both submerged and emerging obstacles are present in open channel, e.g. a grassed waterway.

References

[1] Ogrosky HG, Mockus V. Hydrology of agricultural land. In: Chow VT, editor. Handbook of applied hydrology, vol. 21. McGraw-Hill; 1964. p. 1–97.

[2] Curran JH, Wohl EE. Large woody debris on flow resistance in step-pool channels, Cascade Range, WI. Geomorphology 2003;51:141–57.

[3] Dunkerley D. Organic litter: dominance over stones as a source of interrill flow roughness on low-gradient desert slopes at Fowlers Gap, arid western NSW, Australia. Earth Surf Process Landforms 2003;28:15–29.

[4] Findeling A, Ruy S, Scopel E. Modeling the effects of a partial residue mulch on runoff using a physically based approach. J Hydrol 2003;275:49–66.

[5] Darcy HPG. Recherches expérimentales relatives au movement de l'eau dans les tuyaux. Paris: Imprimerie Nationale; 1857.

[6] Darcy H, Bazin H. Recherches hydrauliques; première partie, recherches experimentales sur l'écoulement de l'eau dans les canaux découverts. Paris: Imprimerie Nationale; 1865.

[7] Weisbach J. Lehrbuch der Ingenieur—und Maschinen—Mechanik. 2nd ed. Braunschweig; 1850.

[8] Chézy A. Rapport sur le canal de l'Yvette. As quoted in Herschel, C. On the origin of the Chézy formula. J Assoc Eng Soc 1897;18:363–5.

[9] Du Buat PLG. Principes d'hydraulique, verifiés par un grand nombre d'expériences faites par ordre du gouvernement. second ed. Paris; 1786.

[10] Prony Rde. Recherches physico-mathématiques sur la théorie des eaux courantes. Paris; 1804.

[11] Darrigol O. Turbulence in 19th-century hydrodynamics. Histor Stud Phys Biol Sci 2002;32:207–62.

[12] Fuentes C, de Leon B, Saucedo H, Parlange J-Y, Antonio ACD. El sistema de ecuaciones de Saint-Venant y Richards del riego por gravedad: 1. La ley potencial de resistencia hidráulica. Ingeniera Hidraulica en México 2004;147(XIX):63–75.

[13] Poiseuille JL. Recherches expérimentales sur le movement des liquides dans les tubes de trés petits diamètres. Comptes rendus de l'Académie des Sciences 1841;12:112–5. 1840:11:961–1048.

[14] Reynolds O. An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous and of the Law of Resistance in parallel channels. Philos Trans Royal Soc London 1883;174.

[15] Manning R. On the flow of water in open channels and pipes. Trans Inst Civil Engr Ireland 1891;20:161–207.

[16] Bélanger J-B. Essai sur la solution numérique de quelques problèmes relatifs au movement permanent des eaux courantes. Paris; 1828.
[17] Barré de Saint Venant AJC. Théorie du movement nonpermanent des eaux avec application aux crues des rivières, et à l'introduction des marées dans leurs lits. Comptes rendus des Séances de l'Académie des Sciences 1871;73:147–54, 237–40.
[18] Boussinesq J. Essai sur la théorie des eaux courantes. Mémoires présentés par divers savants à l'Académie des Sciences 1877;23:1–680.
[19] Dupuit AJ. Traité théorique et pratique de la conduite et de la distribution des eaux. Paris; 1865.
[20] Forchheimer P. Hydraulik, third ed. Leipzig and Berlin; 1930.
[21] Darcy H. Les Fontaines Publiques de la Ville de Dijon. Dalmont, Paris; 1856.
[22] Rouse H, Ince S. History of hydraulics. New York: Dover Pub. Inc; 1963.
[23] Rose CW, Hogarth WL, Ghadiri H, Parlange J-Y, Okom A. Overland flow to and through a segment of uniform resistance. J Hydrol 2002;255:134–50.
[24] Hogarth WL, Parlange J-Y, Rose CW. Addendum to overland flow to and through a segment of uniform resistance. J Hydrol 2003;283:218–24.
[25] Irmay S. On the theoretical derivation of Darcy and Forchheimer formulas. Trans Am Geophys Union 1958;39:702–7.